BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 11, 1956

MATHÉMATIQUE

Un théorème de comportement asymptotique des solutions des systèmes d'équations différentielles

par

F. ALBRECHT

Présenté par T. WAŻEWSKI le 2 Juin 1956

T. Ważewski [4] a donné une méthode topologique pour établir l'existence de solutions asymptotiques d'un système d'équations différentielles,

$$\frac{dx_i}{dt} = f_i(t, x_1, ..., x_n), \qquad i = 1, ..., n, \tag{1}$$

les fonctions f_i étant définies et continues dans un ensemble ouvert D de l'espace euclidien R^{n+1} à $n+1$ dimensions et satisfaisant à des conditions qui assurent l'unicité de l'intégrale dans cet ensemble. Par la suite, cette méthode a pu être généralisée en utilisant différentes notions de rétracte [1], [3], mais en supposant toutefois l'absence de points de glissement intérieur. Le but de cette note est d'établir l'existence d'intégrales asymptotiques dans certaines conditions, permettant de renoncer à cette restriction et applicables surtout aux problèmes concernant les systèmes à petit paramètre.

Soit G un ensemble ouvert, $G \subset D$. Les définitions et notations utilisées seront celles de [1]. Nous dirons qu'un point $p \in \mathrm{Fr}(G) \cap D$ est un point de glissement intérieur de G relatif au système (1), s'il existe un nombre $\varepsilon > 0$ tel que

$$\varphi(p,t) \in G \quad \text{pour} \quad t_p - \varepsilon \leqslant t < t_p \quad \text{et} \quad t_p < t \leqslant t_p + \varepsilon.$$

Désignons pour tout $p \in \bar{G} \cap D$ par $L_0(p)$ et $L_0^+(p)$ la composante de l'ensemble $L(p) \cap \bar{G}$, respectivement $L^+(p) \cap \bar{G}$, qui contient le point p.

THÉORÈME. *Supposons que l'ensemble des points de sortie de G relatifs au système* (1) *est formé par l'ensemble S des points de sortie stricte et par l'ensemble A des points de glissement intérieur, et que l'ensemble A est fermé. Si S est localement connexe et contient une composante S', telle que $L_0(p) \cap A = 0$ pour tout $p \in \bar{S}'$, et s'il existe un ensemble connexe Z ayant*

les propriétés suivantes:

a) $Z \subset G \cup S$, $Z \cap G \neq 0$, $Z \cap S' \neq 0$,

b) $Z \cap S$ *est un rétracte de* S,

c) $Z \cap S$ *n'est pas un rétracte de* Z,

alors il existe un point $p_0 \in Z \cap G$, *tel que* $L^+(p_0) \subset \bar{G}$.

Démonstration. Admettons que $L^+(p) \cap \mathrm{Ext}(G) \neq 0$ pour tout $p \in Z \cap G$. Considérons la transformation $h: Z \to S \cup A$, définie de la façon suivante:

si $p \in Z \cap G$, $h(p)$ est le premier point d'intersection de $L^+(p)$ avec $\mathrm{Fr}(G)$,

si $p \in Z \cap S$, $h(p) = p$ *).

Si $h(Z) \subset S$, cette transformation est continue et notre affirmation résulte immédiatement de la démonstration d'un théorème de Ważewski [4].

Supposons donc qu'il existe un point $p_1 \in Z \cap G$, tel que $h(p_1) \in A$, et considérons l'ensemble

$$Z' = Z \cup L_0^+(p_1),$$

qui est évidemment connexe.

Soit

$$M' = \{p \in Z | h(p) \in S'\}.$$

Cet ensemble est ouvert dans Z'. En effet, si $p \in M' \cap G$, il existe un voisinage W de $h(p)$, tel que $W \cap A = 0$ et un voisinage connexe V de p, tel que $h(V) \subset W$; donc $h(V) \subset S$. La transformation h étant continue sur V, il en résulte que $h(V) \subset S'$.

Si $p \in M' \cap S'$, il existe un voisinage $W \ni h$, tel que $W \cap S \subset S'$ et l'on déduit facilement l'existence d'un voisinage $V \ni p$, tel que $h(V \cap Z') \subset W$.

L'ensemble M' est fermé dans Z'. En effet, soit $p \in \bar{M}' \cap Z'$ et supposons que $h(p) \notin S'$; donc

$$h(p) \in (S - S') \cup A.$$

Si $h(p) \in S - S'$, il existe un voisinage de ce point disjoint de S'; il existe donc un voisinage $V \ni p$, tel que $h(V) \cap S' = 0$, donc $V \cap M' = 0$. Si $h(p) \in A$, soit q le dernier point de $L_0(p)$. On voit facilement que le point q est un point de sortie stricte de G relatif au système (1); il existe donc un $t_0 > t_q$, tel que $\varphi(p,t) \in \mathrm{Ext}(G)$ pour $t_q < t \leqslant t_0$. Comme $L_0(p) \cap S' = 0$ et l'ensemble $C = \{\varphi(p,t) | t_p \leqslant t \leqslant t_0\}$ est compact, il existe un $\varepsilon > 0$, tel que

$$S(C,\varepsilon) \cap \bar{S}' = 0 \text{ **}).$$

---

*) La transformation h est définie en tout point $p \in G \cup S \cup A$, tel que $L^+(p) \not\subset G$.

**) Nous désignons par $S(C,\varepsilon)$ l'ensemble des points m, tels qu'il existe un point $n \in C$ à distance plus petite que ε de m.

En vertu du théorème de continuité des solutions du système (1) par rapport aux conditions initiales, on peut trouver un voisinage $W \ni p$, tel que $h(W) \subset S(C, \varepsilon)$; donc

$$W \cap M' = 0.$$

L'ensemble connexe Z' contient donc un sous-ensemble non vide à la fois ouvert et fermé dans Z', d'où résulte la contradiction qui démontre le théorème.

Il est évident que si l'ensemble G est connexe et satisfait aux conditions du théorème précédent et si S n'est pas un rétracte de $G \cup S$, alors il existe un point $p_0 \in G$ tel que $L^+(p_0) \subset \bar{G}$. Remarquons toutefois que dans ce cas il n'est pas nécessaire de supposer que l'ensemble S est localement connexe et que si $A \neq 0$, l'ensemble S n'est pas connexe.

Remarque 1. Le théorème de [2] est un corollaire immédiat du résultat obtenu ici.

Remarque 2. On peut obtenir un théorème analogue relatif aux demi-intégrales gauches en appliquant la substitution $t = -T$.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE DE LA RÉPUBLIQUE POPULAIRE ROUMAINE

OUVRAGES CITÉS

[1] F. Albrecht, *Remarques sur un théorème de T. Ważewski relatif à l'allure asymptotique des intégrales des équations différentielles*, Bull. Acad. Polon. Sci. Cl. III, **2** (1954), 315.

[2] — *Despre puncte singulare și soluții periodice*, Communicările Acad. R. P. R., **5** (1955), 1035.

[3] A. Pliś, *On a topological method for studying the behaviour of the integrals of ordinary differential equations*, Bull. Acad. Polon. Sci. Cl. III, **2** (1954), 415.

[4] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différientielles ordinaires*, Ann. Soc. Polon. de Math. **20** (1947), 279.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 11, 1956

MATHEMATICS

Generalisation of the Cauchy Problem for a System of Partial Differential Equations

by

A. PLIŚ

Presented by T. WAŻEWSKI on August 8, 1956

In this note we shall deal with the uniqueness (Theorem 1) and the existence (Theorem 2) of the solution $u^1(x, Y), \ldots, u^m(x, Y)$, $Y=(y_1, \ldots, y_n)$ of the system of differential equations

$$u^i_x = f^i(x, Y, u^1, \ldots, u^m, u^i_{y_1}, \ldots, u^i_{y_n}) \quad (i=1, \ldots, m) \tag{1}$$

satisfying the conditions

$$u^i(a_i, Y) = \omega^i(Y) \quad \text{for every } Y \quad (i=1, \ldots, m), \tag{2}$$

where $a_1, \ldots, a_m$ denote constants.

In the particular case $a_1=a_2=\ldots=a_m$ our problem is reduced to the Cauchy problem. The existence of a unique solution of the Cauchy problem for system (1) with the right-hand sides of class C^2 is shown in [4].

THEOREM 1. *If the functions* $f^i(x, Y, p_1, \ldots, p_{m+n})$ $(i=1, \ldots, m)$ *satisfy the inequalities*

$$|f^i(x, Y, p_1, \ldots, p_{m+n}) - f^i(x, Y, \overline{p}_1, \ldots, \overline{p}_{m+n})| \leqslant B \sum_{j=1}^{m} |p_j - \overline{p}_j| + C \sum_{j=m+1}^{m+n} |p_j - \overline{p}_j| \tag{3}$$

for $|x| \leqslant \beta$ *and for arbitrary* $Y, p_1, \ldots, p_{m+n}$, $\overline{p}_1, \ldots, \overline{p}_{m+n}$, *and the system of numbers* $a_1, \ldots, a_m$ *satisfies the inequalities*

$$|a_i| \leqslant \beta, \tag{4}$$

where B, β $(\beta > 0)$ *are constants satisfying the inequality*

$$2mB\beta < 1, \tag{5}$$

then there exists at most one bounded solution *) $u^1(x, Y), \ldots, u^m(x, Y)$ *of system* (1), *defined for* $|x| \leqslant \beta$, *having a complete differential for* $|x| \leqslant \beta$ *and satisfying conditions* (2).

*) Besides the unique bounded solution many unbounded solutions may exist.

To begin with we shall prove a certain inequality (inequality (7)), which will also be utilised in the proof of Theorem 2.

Let Γ be the space of (vector) functions $U(x, Y)=(u^1(x, Y), \ldots, u^m(x, Y))$ defined, continuous and bounded for

$$|x| \leqslant \beta, \quad \text{with the norm} \quad \|U\| = \sup_{(|x| \leqslant \beta, i=1,\ldots,m)} |u^i(x, Y)|.$$

The solution $z(x, Y)$ of the differential equation

$$z_x = f^i(x, Y, U(x, Y), z_{y_1}, \ldots, z_{y_n}), \tag{6}$$

where $U(x,Y)$ is a given function from space Γ and i a fixed index $(1 \leqslant i \leqslant m)$, satisfying the initial condition $z(a_i, Y)=\omega^i(Y)$, depends on the index i and on the choice of the function U. Let us therefore denote it by $z^i(x, Y; U)$ provided such a solution exists and has a complete differential for $|x| \leqslant \beta$.

It can easily be observed that the function $v(x, Y)=z^i(x, Y; U)-z^i(x, Y; \bar{U})$, where $U, \bar{U}$ are elements of space Γ for which there exist suitable solutions of equation (6), satisfies the differential inequality $|v_x| \leqslant mB\|U-\bar{U}\| + C\sum_{j=1}^{n} |v_{y_j}|$ and the identity $v(a_i, Y) \equiv 0$. Consequently the theorem on the partial differential inequality (cf. [3] théorème 2, p. 17 for $k=1$, $N>0$, $N \to 0$) implies the inequality $|v(x, Y)| \leqslant mB\|U-\bar{U}\||x-a_i|$. Thus the operation $Z=Z(U): z^i=z^i(x, Y; U)$ $(i=1, \ldots, m)$ satisfies the inequality

$$\|Z(U)-Z(\bar{U})\| \leqslant 2mB\beta\|U-\bar{U}\| \quad \text{for any} \quad U, \bar{U} \tag{7}$$

for which it is defined.

For the proof of Theorem 1 it will be observed that any bounded solution U of system (1) and conditions (2) satisfies the equation

$$U=Z(U). \tag{8}$$

By (7) and (5) at most one such solution can exist.

THEOREM 2. *Suppose that for* $|x| \leqslant a (a>0)$ *the functions* $f^i(x, r_1, \ldots, r_{m+2n})$ $(i=1, \ldots, m)$ *together with their derivatives of the first order with respect to the variables* $r_1, \ldots, r_{m+2n}$ *are continuous and satisfy the inequalities* $|f^i| \leqslant M$,

$$|f^i_{r_j}| \leqslant A \quad (j=1, \ldots, n), \quad |f^i_{r_j}| \leqslant B \quad (j=n+1, \ldots, n+m), \quad |f^i_{r_j}| \leqslant C \quad (j=m+n+1, \ldots, m+2n) \tag{9}$$

and

$$|f^i_{r_j}(x, r_1, \ldots, r_{m+2n}) - f^i_{r_j}(x, \bar{r}_1, \ldots, \bar{r}_{m+2n})| \leqslant L\sum_{k=1}^{m+2n} |r_k - \bar{r}_k| \tag{10}$$

for any $r_1, \ldots, r_{m+2n}, \bar{r}_1, \ldots, \bar{r}_{m+2n}$ $(j=1, \ldots, m+2n)$, *where* a, M, A, B, C, L *are constants.*

Let the functions $\omega^i(Y)$ $(i=1, \ldots, m)$ *together with their derivatives of the first order with respect to* $y_1, \ldots, y_n$ *be continuous in* E^n *and satisfy the inequalities*

$$|\omega^i(Y)| \leqslant N, \quad |\omega^i_{y_j}(Y)| \leqslant H \quad (j=1,\ldots,n) \quad \text{and} \tag{11}$$

$$|\omega^i_{y_j}(y_1,\ldots,y_n) - \omega^i_{y_j}(\bar{y}_1,\ldots,\bar{y}_n)| \leqslant J\sum_{k=1}^{n}|y_k-\bar{y}_k|$$

for any $y_1,\ldots,y_n$, $\bar{y}_1,\ldots,\bar{y}_n$ $(j=1,\ldots,n)$.

Under the above assumptions for every system of numbers $a_1,\ldots,a_m$ *satisfying inequality* (4) *for*

$$\beta = \min\left(a, \frac{1}{4n(1+nJ)\,[(1+mT)^2L + mB(1+2J)]}\right),$$

where $T=2H+A\min(4a, 1/B+L)$, *there exists exactly one solution* $u^1(x,Y),\ldots,u^m(x,Y)$ *of system* (1) *satisfying conditions* (2), *defined, bounded and of class* C^1 *for* $|x|\leqslant\beta$. *Moreover this solution satisfies the conditions*

$$|u^i_{y_j}(x,Y)| \leqslant T \quad (i=1,\ldots,m, \quad j=1,\ldots,n), \tag{12}$$

$$|u^i_{y_j}(x,y_1,\ldots,y_n) - u^i_{y_j}(x,\bar{y}_1,\ldots,\bar{y}_n)| \leqslant (2J+1)\sum_{k=1}^{n}|y_k-\bar{y}_k| \tag{13}$$

for any $y_1,\ldots,y_n$, $\bar{y}_1,\ldots,\bar{y}_n$ $(i=1,\ldots,m,\ j=1,\ldots,n)$, $|x|\leqslant\beta$.

For the proof let us consider a subspace S of space Γ consisting of functions $U(x,Y)$ having continuous derivatives with respect to $y_1,\ldots,y_n$ and satisfying inequalities (12), (13) and the inequality

$$|u^i_{y_j}(x,Y) - u^i_{y_j}(\bar{x},Y)| \leqslant [A+mBT+nC(2J+1)]|x-\bar{x}| \quad \text{for} \quad |x|\leqslant\beta,\ |\bar{x}|\leqslant\beta \tag{14}$$

and arbitrary Y $(i=1,\ldots,m,\ j=1,\ldots,n)$. It is easy to verify that space S is complete. For each U of the space S the operation $Z(U)$ is defined and the function $Z{:}Z=Z(U)$ is of class C^1 for $|x|\leqslant\beta$ and satisfies inequality (13) ([1] Theorem 2). Making use of the theory of characteristics (see e. g. [2], p. 420, point 2) we can prove that

$$|z^i_{y_j}(x,Y;U)| \leqslant H + (A+mBT)|x-a_i| \quad \text{for} \quad |x|\leqslant\beta$$

and arbitrary Y, whence it follows that function Z satisfies inequality (12). It is easy to verify that it also satisfies inequality (14) and that it is bounded. Thus we have shown that the transformation $\bar{Z}=Z(U)$ transforms the complete space S into itself. Besides, it satisfies condition (7) with constants obviously satisfying condition (5). Consequently, as we know, equation (8) has a solution in the space S, which can be obtained by the method of successive approximation starting from an arbitrary function of space S. By Theorem 1 this solution is unique. It can easily be verified that this solution satisfies Theorem 2.

Remark. The following variant of problem (1), (2) can be formulated: Given functions $\varphi_i(Y)$ $(i=1,...,m)$ defined in a certain n-dimensional bounded and closed domain D and satisfying the relations $\varphi_1(Y)=\varphi_2(Y)=...=\varphi_m(Y)$ on its boundary. Functions $\omega^i(Y)$ $(i=1,...,m)$ are defined in D and functions f^i in a suitable bounded $(2n+m+1)$-dimensional domain Ω. We seek the solution U of system (1) defined in a suitable $(n+1)$-dimensional bounded domain Δ (e. g., $D:|y_i|\leqslant 1$; $\Delta:|x|\leqslant\varepsilon$, where ε is a positive number, $|y_i|\leqslant 1-C|x|$, $i=1,...,n$; $\Omega=\Delta\times R^{n+m}$) and satisfying the conditions

$$u^i(\varphi_i(Y),Y)=\omega^i(Y) \quad \text{in} \quad D \qquad (i=1,...,m).$$

For the existence of a unique solution of this problem we must make assumptions analogous to those of Theorem 2, — assumptions similar to those regarding functions ω^i for functions φ_i, and the assumption that the areas $x=\varphi_i(Y)$ are contained in Δ.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] A. Pliś, *On the estimation of the existence domain for solutions of a non-linear partial differential equation of the first order*, Bull. Acad. Polon. Sci., Cl. III, **4** (1956), 3, 125.

[2] — *Characteristics of non-linear partial differential equations*, Bull. Acad. Polon. Sci., Cl. III, **2** (1954), 9, 419.

[3] J. Szarski, *Sur certains systèmes d'inégalités différentielles aux dérivées partielles du premier ordre*, Annales de la Soc. Polon. Math. **21** (1948), 1.

[4] T. Ważewski, *Sur le problème de Cauchy relatif à un système d'équations aux dérivées partielles*, Annal. de la Soc. Polon. Math. **15** (1936), 101.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 11, 1956

MATHEMATICS

On the Accuracy of Approximation with Nodes

by

S. PASZKOWSKI

Presented by H. STEINHAUS on August 9, 1956

1. Let $\mathcal{C}$ denote the class of continuous functions in the closed interval $I=\langle a,b\rangle$ with the norm $\|\xi\|=\max_{t\in I}|\xi(t)|$, and $\mathcal{W}_n$ the class of all algebraic polynomials of at most n-th degree. In the theory of uniform approximation in the ordinary sense we introduce the functional

$$\varepsilon_n(\xi)=\inf_{\Psi\in\mathcal{W}_n}\|\xi-\Psi\| \qquad (\xi\in\mathcal{C}) \tag{1}$$

expressing the error of the best approximation (in the sense of the introduced norm) of the continuous function ξ by polynomials of at most n-th degree. In *the approximation with nodes* ([1]-[3]), $t_1,t_2,\ldots,t_m$, where $a\leqslant t_1<t_2<\ldots<t_m\leqslant b$ of function ξ we confine ourselves to polynomials ω of class $\mathcal{W}_n$ $(n>m)$ such that

$$\omega(t_k)=\xi(t_k) \qquad \text{for} \qquad k=1,2,\ldots,m. \tag{2}$$

The set of these polynomials is denoted by $\mathcal{W}_n(\xi;\,T)$, where $T=\{t_1,t_2,\ldots,t_m\}$. A functional that is analogous to (1) is connected with the type of approximation:

$$\varepsilon_n(\xi;\,T)=\inf_{\omega\in\mathcal{W}_n(\xi;T)}\|\xi-\Psi\| \qquad (\xi\in\mathcal{C}).$$

2. The properties of approximation with nodes can be compared with the properties of ordinary uniform approximation. In particular, we can examine (for $\varepsilon_n(\xi)>0$) the quotient $\varepsilon_n(\xi;\,T)/\varepsilon_n(\xi)$ indicating how many times the error of the best approximation is increased by the imposition of conditions (2) on the approximating polynomials (since $\mathcal{W}_n(\xi;\,T)\subset\mathcal{W}_n$, it is obvious that for every function $\xi\in\mathcal{C}$ and system of nodes T we have $\varepsilon_n(\xi;\,T)\geqslant\varepsilon_n(\xi)$). In papers [2] and [3] we deal with the *a priori* estimate of the quotient $\varepsilon_n(\xi;\,T)/\varepsilon_n(\xi)$, independent of function ξ. In paper [3] it has been proved (theorem 5.2) that there *exists a natural number* $\nu=\nu(T)$ *such that for every function* $\xi\in\mathcal{C}$ *and every* $n\geqslant\nu$ *we have*

$$\varepsilon_n(\xi;\,T)<2\varepsilon_n(\xi). \tag{3}$$

Number $\nu(T)$, however, was not given explicitely in [3]. It is given in this paper, to wit, the following theorem holds:

THEOREM. *If* $m \geqslant 3$ *and* $n \geqslant 14[p/c]+12$, *where*

$$p = \min\left\{6(b-a), \frac{m-1}{\pi}(2d-(m-1)c)\right\},$$

$$c = \tfrac{1}{2}\min_{1\leqslant k\leqslant m-1}(t_{k+1}-t_k), \qquad d = \max_{1\leqslant k\leqslant m}\max\{t_k - a, b-t_k\},$$

and $[p/c]$ *denotes the integral part of the number* p/c, *then inequality* (3) *is satisfied for every function* $\xi \in \mathcal{C}$.

Proof. It is known that

(I) There exists a natural number $\nu = \nu(T)$ (having the same meaning as in theorem 5.2, quoted above, of paper [3]) such that for any numbers $x_1, x_2, \ldots, x_m$ there exists a polynomial φ satisfying the conditions $1^\circ\ \varphi \in \mathcal{W}_\nu$, $2^\circ\ \varphi(t_k) = x_k$ for $k = 1, 2, \ldots, m$, $3^\circ\ \|\varphi\| = \max_i |x_i|$. This is theorem 5.1 of paper [3], which easily implies the inequality (3) for $n \geqslant \nu(T)$. Therefore we shall undertake the construction of polynomial φ and an upper estimate of its degree.

(II) If $n > p/c$, then for any numbers $y_1, y_2, \ldots, y_m$ there exists a polynomial χ satisfying conditions $1^\circ\ \chi \in \mathcal{W}_n$, $2^\circ\ \chi(t_k) = y_k$ for $k = 1, 2, \ldots, m$, $3^\circ\ \|\chi\| \leqslant \frac{cn+p}{cn-p}\max_i |y_i|$. This theorem is obtained by a slight modification of the proof of theorem 6.4 of paper [3]; we omit the details.

It will also be observed that the constants defined in the assumptions of the theorem in question satisfy the inequalities

$$c \leqslant \frac{1}{2}(t_{k+1}-t_k) \quad \text{for} \quad k = 1, 2, \ldots, m-1, \quad \text{whence} \quad c \leqslant \frac{1}{2(m-1)}(t_m - t_1),$$

$$d \geqslant t_m - a \geqslant t_m - t_1, \quad \text{whence} \quad 2d-(m-1)c \geqslant \frac{3}{2}(t_m - t_1),$$

$$p \geqslant \min\left\{6(t_m-t_1), \frac{m-1}{\pi}\frac{3}{2}(t_m-t_1)\right\}, \quad p/c \geqslant 2(m-1)\min\left\{6, \frac{3(m-1)}{2\pi}\right\}.$$

Since the right side of the last inequality is an increasing function of m, we have for $m \geqslant 3$

$$p/c \geqslant 4\min\left\{6, \frac{6}{2\pi}\right\} = \frac{12}{\pi}, \quad [p/c] \geqslant 3. \tag{4}$$

Proceeding to the proof of the theorem we assume that

$$n > p/c \tag{5}$$

and define l as the least natural number satisfying the inequality

$$\sin \pi/2(2l+1) \leqslant (cn-p)/(cn+p). \tag{6}$$

Let us consider the Tchebyshev polynomial $T_{2l+1}(t)=\cos(2l+1)\arccos t$. It is easy to verify that with t changing from $-\sin\pi/2(2l+1)$ to $\sin\pi/2(2l+1)$ this polynomial increases from -1 to $+1$ or decreases from $+1$ to -1 (according to the value of l). Hence and from inequality (6) it follows that with t running over the interval $\langle-(cn-p)/(cn+p),(cn-p)/(cn+p)\rangle$ the polynomial $T_{2l+1}(t)$ takes on all values from -1 to $+1$. Therefore for any numbers $x_1,x_2,\ldots,x_m$ numbers $y_1,y_2,\ldots,y_m$ can be found such that

$$T_{2l+1}(Y_k)=x_k/\max_i|x_i| \quad (k=1,2,\ldots,m), \tag{7}$$

$$|y_k|\leqslant(cn-p)/(cn+p) \quad (k=1,2,\ldots,m. \tag{8}$$

By assumption (5) and inequality (8) the theorem quoted in (II) implies the existence of a polynomial χ such that 1° $\chi\in\mathcal{W}_n$, 2° $\chi(t_k)=y_k$ for $k=1,2,\ldots,m$, 3° $\|\chi\|\leqslant\dfrac{cn+p}{cn-p}\max_k|y_k|\leqslant 1$.

If we assume that $\varphi(t)=\max_i|x_i|\cdot T_{2l+1}(\chi(t))$, this polynomial will satisfy the conditions of Theorem (I), namely 1° $\varphi\in\mathcal{W}_{(2l+1)n}$, 2° $\varphi(t_k)=x_k$ for $k=1,2,\ldots,m$ by equality (7) and condition 2° of Theorem (II), 3° $\|\varphi\|=\max_k|x_k|$ since we have proved that for $t\in I$ we have $|\chi(t)|\leqslant 1$, and it is known that for $|h|\leqslant 1$ we have $|T_{2l+1}(h)|\leqslant 1$.

We have thus obtained the inequality $\nu(T)\leqslant(2l+1)n$. From the definition of number l it follows that

$$\sin\frac{\pi}{2(2l-1)}>\frac{cn-p}{cn+p},\qquad 2l-1<\frac{\pi}{2}\cdot\frac{cn+p}{cn-p},\qquad \nu(T)<\left(\frac{\pi}{2}\cdot\frac{cn+p}{cn-p}+2\right)n. \tag{9}$$

Under assumption (5) the right side of the inequality thus obtained has its minimum for $n=\left(1+\sqrt{\dfrac{2\pi}{\pi+4}}\right)p/c\approx 1{,}938\,p/c$. Since, however, n is to be a natural number, we can use this information only approximately. We take $n=2[p/c]+1$. Assumption (5) is then satisfied since, in view of the inequality $[p/c]>p/c-1$, it holds if $2(p/c-1)+1\geqslant p/c$, i. e. $p/c>1$, which is true by (4). For such an n we obtain from (9)

$$\nu(T)<\left(\frac{\pi}{2}\cdot\frac{n+p/c}{n-P/c}+2\right)n<\left(\frac{\pi}{2}\cdot\frac{n+[p/c]+1}{n-[p/c]-1}+2\right)n$$

$$=\left(\frac{\pi}{2}\cdot\frac{3[p/c]+2}{[p/c]}+2\right)(2[p/c]+1)=(3\pi+4)[p/c]+\frac{7}{2}\pi+2+\pi/[p/c].$$

Using inequality (4) twice we obtain

$$\nu(T)<(3\pi+4)[p/c]+\frac{23}{6}\pi+2=$$

$$=(14-(10-3\pi))[p/c]+\frac{23}{6}\pi+2\leqslant 14[p/c]+\frac{77}{6}\pi-28.$$

Since $\frac{77}{6}\pi - 28 = 12{,}327\ldots$, and we estimate an integer *a priori*, we obtain

$$\nu(T) \leqslant 14[p/c] + 12$$

and the theorem is proved.

3. Finally it will be observed that

1° In the theorem here proved we have confined ourselves to the values $m \geqslant 3$ since for $m = 1$ inequality (3) holds for every $n > m$ [3] and for $m = 2$ we formerly obtained strong results ([3], Theorem 7.10);

2° in the proof of the theorem in question the interesting fact is that we actually make use of a theorem that is only slightly weaker than inequality (3):

$$\varepsilon_n(\xi;\ T) \leqslant \frac{2}{1 - p/cn}\varepsilon_n(\xi) \qquad \text{for} \qquad n > p/c$$

([3], Theorem 6.4);

3⁰ the *a priori* estimate of number $\nu(T)$ such that for $n \geqslant \nu(T)$ and for any function $\xi \in \mathcal{C}$ inequality (3) holds, can also be obtained by a different, longer method, by supplementing Wolibner's theorem [4] with an *a priori* estimate of the degree of the polynomial with which this theorem is concerned. That method, however, seems to lead to less satisfactory results than those presented here.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] S. Paszkowski, *Sur l'approximation uniforme avec des noeuds*, Annales Polonici Mathematici **2** (1955), 118.

[2] — *On the Weierstrass approximation theorem*, Colloqium Mathematicum **4** (1956). (in preparation).

[3] — *On the approximation with nodes*, Rozprawy Matematyczne, Instytutu Matematycznego PAN, (in preparation).

[4] W. Wolibner, *Sur un polynôme d'interpolation*, Colloquium Mathematicum **2** (1941-1951), 136-137.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 11, 1956

MATHEMATICS

Sets Filled by Asymptotic Integrals of Ordinary Differential Equations

by

A. PLIŚ

Presented by T. WAŻEWSKI on August 12, 1956

This note presents a method of proving certain properties of sets filled by asymptotic integrals of ordinary differential equations. The method is used to show that the sets in question are graphs of functions satisfying the Lipschitz condition (Theorem 2). A more detailed theorem on the subject has been given by Z. Szmydt in paper [2] *).

1. Hypothesis H. *Suppose that the right-hand members of a system of $k+m$ ordinary differential equations (written in the vector form):*

$$x'=f(x,y,t), \quad y'=g(x,y,t), \tag{1}$$
$$x=(x_1,\ldots,x_k), \quad y=(y_1,\ldots,y_m), \quad f=(f^1,\ldots,f^k), \quad g=(g^1,\ldots,g^m),$$

are continuous in a certain open set S, through each point of which passes exactly one integral of system (1), and that the functions f,g, for $(x,y,t)\in S$, $(\bar{x},\bar{y},t)\in S$, satisfy inequalities

$$(f(x,y,t)-f(\bar{x},\bar{y},t))(x-\bar{x})\leqslant\alpha(t)(x-\bar{x})^2 \quad \text{for} \quad |y-\bar{y}|=|x-\bar{x}| \text{ **}), \tag{2}$$

$$(g(x,y,t)-g(\bar{x},\bar{y},t))(y-\bar{y})\geqslant\alpha(t)(y-\bar{y})^2 \quad \text{for} \quad |x-\bar{x}|\leqslant|y-\bar{y}|, \tag{3}$$

where $\alpha(t)$ is a function summable in every finite interval.

THEOREM 1. *Assume hypothesis H. If $x(t)$, $y(t)$ and $\bar{x}(t)$, $\bar{y}(t)$ are integrals of system (1), $(x(t),y(t))\in S$, $(\bar{x}(t),\bar{y}(t))\in S$ for $a\leqslant t<b$ and*

$$|x(a)-\bar{x}(a)|\leqslant|y(a)-\bar{y}(a)|, \tag{4}$$

then

$$|y(t)-\bar{y}(t)|\geqslant\exp\Big(\int_a^t\alpha(s)\,ds\Big)|y(a)-\bar{y}(a)| \quad \text{for} \quad a\leqslant t<b. \tag{5}$$

*) Paper [2] concerns only integrals tending to a singular point. Z. Szmydt's results concerning sets filled by integrals satisfying more general conditions have not yet been published.

**) $|z|=\sqrt{z^2}$.

Moreover

$$|x(t)-\bar{x}(t)| \leqslant |y(t)-\bar{y}(t)| \quad \text{for} \quad a \leqslant t < b. \tag{6}$$

THEOREM 1′. *Assume inequality* (3) *and the inequality*

$$(f(x,y,t)-f(\bar{x},\bar{y},t))(x-\bar{x}) < \alpha(t)(x-\bar{x})^2 \quad \text{for} \quad |y-\bar{y}| = |x-\bar{x}| > 0 \tag{7}$$

in a certain set Z. *If* $x(t), y(t)$ *and* $\bar{x}(t), \bar{y}(t)$ *are integrals of system* (1), $(x(t), y(t)) \in Z$, $(\bar{x}(t), \bar{y}(t)) \in Z$ *for* $a \leqslant t < b$ *and*

$$|x(a)-\bar{x}(a)| < |y(a)-\bar{y}(a)|, \tag{8}$$

then inequalities (5) *and* (6) *are satisfied for* $a \leqslant t < b$.

We shall confine ourselves to the proof of Theorem 1′ since Theorem 1 can easily be obtained from it by approximating system (1) by systems $x' = f - \sigma x$, $y' = g$ $(\sigma > 0)$ satisfying conditions (3), (7).

To prove the theorem indirectly, suppose that inequalities (5) and (6) do not hold in the whole interval $a \leqslant t < b$. Denote by c, $a \leqslant c < b$, the greatest number for which inequalities (5), (6) are satisfied for $a \leqslant t \leqslant c$. We shall prove that inequality (6) holds in a certain right-hand neighbourhood of c. If $|x(c)-\bar{x}(c)| < |y(c)-\bar{y}(c)|$, this follows from the continuity of functions $x, \bar{x}, y, \bar{y}$. If $|x(c)-\bar{x}(c)| = |y(c)-\bar{y}(c)|$, then by (1), (7) and the validity of (5) for $t = c$

$$\frac{d}{dt}(x(t)-\bar{x}(t))^2 < 2\alpha(t)(x(t)\theta-\bar{x}(t))^2 \quad \text{for} \quad t = c, \quad \text{and by} \quad (1), (3)$$

$$\frac{d}{dt}(y(t)-\bar{y}(t))^2 \geqslant 2\alpha(t)(y(t)-\bar{y}(t))^2 = 2\alpha(t)(x(t)-\bar{x}(t))^2 \quad \text{for} \quad t = c.$$

Thus it follows that $d/dt(y(t)-\bar{y}(t))^2 > d/dt(x(t)-\bar{x}(t))^2$ for $t = c$. Inequality (6) is thus satisfied in a certain right-hand neighbourhood of c. In that neighbourhood the function $h = (y(t)-\bar{y}(t))^2$ satisfied the differential inequality $h' \geqslant 2\alpha(t)h$. Since inequality (5) is satisfied in c, it follows that it is also satisfied in a certain right-hand neighbourhood of c. This contradiction of the definition of c completes the proof of the theorem.

THEOREM 2. *If the assumption* H *is satisfied, the set* U *of points* (X, Y, T) *lying on the integrals of* (1) *remaining in* S *for* $T \leqslant t < \infty$ *and satisfying the condition*

$$\lim_{t\to\infty} |y(t)| \exp\left(-\int_0^t \alpha(s)\,ds\right) = 0 \tag{9}$$

is a graph of a single-valued function $Y = \psi(X, T)$ *defined in a certain set* A $(A \subset E^{k+1})$ *satisfying the Lipschitz condition with respect to all the variables and in particular the condition*

$$|\psi(x,t) - \psi(\bar{x},t)| \leqslant |x-\bar{x}| \tag{10}$$

if the set A *or the set* U *is an empty set.*

Suppose that the set U is neither a graph of a function satisfying condition (10) nor an empty set. In this case there exist two points, (X, Y, T), and $(\overline{X}, \overline{Y}, T)$, satisfying the condition

(11) $\quad |X-\overline{X}| < |Y-\overline{Y}|$ and such that the integral $x(t), y(t)$ passing through X, Y, T and the integral $\overline{x}(t), \overline{y}(t)$ passing through $\overline{X}, \overline{Y}, T$ are defined in the interval $T \leqslant t < \infty$ and satisfy (9).

Hence it follows that

$$\lim_{t\to\infty} |y(t)-\overline{y}(t)| \exp\left(-\int_0^t a(s)\,ds\right) = 0.$$

By (11), Theorem 1 implies

$$|y(t)-\overline{y}(t)| \exp\left(-\int_T^t a(s)\,ds\right) \geqslant |Y-\overline{Y}|.$$

Consequently $|Y-\overline{Y}| = 0$. Since this contradicts inequality (11), it is proved that the set U is a graph of a function satisfying condition (10). The Lipschitz condition with respect to t follows from the continuity of the right-hand sides of system (1) and condition (10).

The proof of Theorem 2 is thus completed *).

Remark 1. If system (1) satisfies assumptions H in the neighbourhood of the set $Z: |x| \leqslant 1$, $|y| \leqslant 1$, $0 \leqslant t < \infty$ and moreover $\int_0^\infty a(s)\,ds = \infty$,

$$f(x,y,t)x < 0 \quad \text{for} \quad |x| = 1, \ |y| \leqslant 1, \quad 0 \leqslant t < \infty,$$
$$g(x,y,t)y > 0 \quad \text{for} \quad |y| = 1, \ |x| \leqslant 1, \quad 0 \leqslant t < \infty$$

then the set U of points (X, Y, T) lying on the integrals remaining in the set Z for $T \leqslant t < \infty$ is the graph of the function

$$\psi(X, T) \quad \text{defined in the whole set} \quad |X| \leqslant 1, \quad 0 \leqslant T < \infty.$$

The fact that for each pair $\widetilde{x}, \widetilde{t}$ ($|\widetilde{x}| \leqslant 1$, $0 \leqslant \widetilde{t} < \infty$) there exists a $\widetilde{y}$ ($|\widetilde{y}| < 1$) such that the integral passing through the point $(\widetilde{x}, \widetilde{y}, \widetilde{t})$ remain in the set Z, can be proved with the aid of T. Ważewski's theorem ([3] Théorème 1 p. 299).

Remark 2. Theorem 2 may also be applied to the system, dealt with in paper [1], of two differential equations of the form $x' = A(x,y)$, $y' = y + B(x,y)$, where functions A, B are of class C^1 in a certain neighbourhood of the point $x=0$, $y=0$ and vanish, together with its derivatives of the first order, at point $x=0$, $y=0$, if we assume in Theorem 2 that $k=1$, $m=1$, $f=A$, $g=y+B$, $a(t)=\varepsilon(0<\varepsilon<1)$, $S=\omega\times R$, ω being a neighbourhood of point $x=0$, $y=0$ chosen so that relations (2) and (3) hold, and R denoting the axis t.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

*) The author owes to Z. Szmydt the idea of applying the method here presented to systems dependent on time.

REFERENCES

[1] K. A. Keil, *Das qualitative Verhalten der Integralkurven einer gewöhnlichen Differentialgleichung erster Ordnung in der Umgebung eines singulären Punktes*, Jahresbericht Deutsch. Math. Verein. **57** (1955), 111.

[2] Z. Szmydt, *Sur la structure de l'ensemble engendré par les intégrales tendant vers le point singulier du système d'équations différentielles*, Bull. Acad. Polon. Sci. Cl. III, **1** (1953), 223.

[3] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. **20** (1947), 279.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 11, 1956

MATHÉMATIQUE

Sur une mesure de la dispersion d'une distribution de directions

par

J. SIMAIKA

Présenté par H. STEINHAUS le 10 Septembre 1956

Introduction

Le problème de déterminer les valeurs caractéristiques d'une distribution de directions se complique du fait que la valeur de la variable ne change pas d'une quantité constante quand l'origine est déplacé. En effet, si l'on représente la direction $\theta(0 \leqslant 0 < 2\pi)$ par un point de la circonférence du cercle unitaire et si θ_0 et θ_α désignent la valeur de cette direction quand l'origine est aux points 0 et α respectivement, l'on voit que

$$(1) \qquad \begin{aligned} \theta_\alpha &= \theta_0 - \alpha \qquad \text{pour} \quad \alpha \leqslant \theta_0 < 2\pi, \\ \theta_\alpha &= \theta_0 - \alpha + 2\pi \qquad \text{pour} \quad 0 \leqslant \theta_0 < \alpha. \end{aligned}$$

L'addition de directions sera dans la suite entendue modulo 2π pour avoir, comme valeur θ d'une direction, toujours un nombre de l'intervalle $(0 \leqslant \theta < 2\pi)$.

Si l'on considère, maintenant, une direction aléatoire Θ définie sur le cercle unitaire $(0 \leqslant \theta < 2\pi)$ et ayant la loi de distribution $F(\theta) = P_r \cdot [0 \leqslant \Theta < \theta]$, on peut voir que les notions de moyenne,

$$[1] \qquad \mathfrak{M}\Theta = \int_0^{2\pi} \theta \, dF(\theta), \quad \text{et de variance} \quad V\Theta = \int_0^{2\pi} (\theta - \mathfrak{M}\Theta)^2 \, dF(\theta), \; ^*)$$

ne sont plus valables; car, n'étant plus invariantes, elles peuvent prendre des valeurs arbitraires par un choix approprié de l'origine.

Pour éviter cette difficulté, nous avons été amenés à considérer des fonctions périodiques de la variable ce qui nous a permis de donner dans la note [1] une valeur typique $\mathfrak{T}\Theta$ définie par la relation:

$$(2) \qquad \frac{\sin \mathfrak{T}\Theta}{\mathfrak{M} \sin \Theta} = \frac{\cos \mathfrak{T}\Theta}{\mathfrak{M} \cos \Theta} = \frac{1}{+\sqrt{\mathfrak{M}^2 \sin \Theta + \mathfrak{M}^2 \cos \Theta}}$$

qui remplace avantageusement la valeur moyenne et qui a un certain nombre de propriétés désirables.

Dans ce qui suit nous étudions une mesure de la dispersion de la distribution.

*) Intégrale de Stieltjes-Riemann.

Définition et propriétés

Soit Θ une variable aléatoire définie sur le cercle unitaire $(0 \leqslant \theta < 2\pi)$ et soit $F(\theta)$ sa loi de distribution. Nous suggérons comme mesure de la dispersion la quantité $\mathfrak{D}\Theta$ définie par la relation:

$$\cos \mathfrak{D}\Theta = +\sqrt{\mathfrak{M}^2 \sin\Theta + \mathfrak{M}^2 \cos\Theta}. \tag{3}$$

$\mathfrak{D}\Theta$ est la valeur principale positive de l'arc cosinus $\sqrt{\mathfrak{M}^2 \sin\Theta + \mathfrak{M}^2 \cos\Theta}$.

Cette mesure de la dispersion a les propriétés suivantes:

1. $\mathfrak{D}\Theta$ est un arc du cercle unitaire.

2. $\mathfrak{D}\Theta$ existe toujours quelle que soit la loi de distribution. En effet, $\cos \mathfrak{D}\Theta = |\varphi(1)|$, où $\varphi(t)$ est la fonction caractéristique de la loi de distribution, fonction dont le module est toujours inférieur ou égal à l'unité.

3. $0 \leqslant \mathfrak{D}\Theta \leqslant \frac{\pi}{2}$.

$\mathfrak{D}\Theta$ prend la valeur 0 si et seulement si la variable Θ est p. s. certaine.
$\mathfrak{D}\Theta$ prend la valeur $\pi/2$ si et seulement si $\mathfrak{M}\sin\Theta = \mathfrak{M}\cos\Theta = 0$ *).

4. $\mathfrak{D}\Theta$ est indépendante de l'origine choisie. En effet,

$$\begin{aligned}\cos \mathfrak{D}(\Theta+\alpha) &= \sqrt{\mathfrak{M}^2 \sin(\Theta+\alpha) + \mathfrak{M}^2 \cos(\Theta+\alpha)} \\ &= \sqrt{\mathfrak{M}^2 \sin\Theta + \mathfrak{M}^2 \cos\Theta} \\ &= \cos \mathfrak{D}\Theta.\end{aligned}$$

5. $\log \cos \mathfrak{D}\Theta$ est additif dans le cas d'indépendance. En effet,

$$\begin{aligned}\cos \mathfrak{D}(\Theta_1+\Theta_2) &= \sqrt{\mathfrak{M}^2 \sin(\Theta_1+\Theta_2) + \mathfrak{M}^2 \cos(\Theta_1+\Theta_2)} \\ &= \sqrt{(\mathfrak{M}\sin\Theta_1\cos\Theta_2 + \mathfrak{M}\sin\Theta_2\cos\Theta_1)^2 + \\ &\quad + (\mathfrak{M}\cos\Theta_1\cos\Theta_2 - \mathfrak{M}\sin\Theta_1\sin\Theta_2)^2},\end{aligned}$$

et si les deux variables Θ_1 et Θ_2 sont indépendantes

$$\begin{aligned}\cos \mathfrak{D}(\Theta_1+\Theta_2) &= \sqrt{(\mathfrak{M}^2\sin\Theta_1 + \mathfrak{M}^2\cos\Theta_1)(\mathfrak{M}^2\sin\Theta_2 + \mathfrak{M}^2\cos\Theta_2)} \\ &= \cos \mathfrak{D}\Theta_1 \cdot \cos \mathfrak{D}\Theta_2.\end{aligned}$$

6. Quand la corde joignant les deux points Θ_1 et Θ_2 de la circonférence est prise comme distance entre les deux points, le minimum de $\mathfrak{M}|e^{i\theta} - e^{i\Theta}|^2$, valeur moyenne du carré de cette distance, est atteint quand $\theta = \mathfrak{T}\Theta$ **). Le minimum de cette distance (corde) sous-tend précisément l'arc $\mathfrak{D}\Theta$. En effet, la corde minima

$$\begin{aligned}\sqrt{\mathfrak{M}|e^{i\mathfrak{T}\Theta} - e^{i\Theta}|^2} &= \sqrt{\mathfrak{M}(\cos\mathfrak{T}\Theta - \cos\Theta)^2 + \mathfrak{M}(\sin\mathfrak{T}\Theta - \sin\Theta)^2} \\ &= \sqrt{2(1 - \mathfrak{M}\cos\mathfrak{T}\Theta\cos\Theta - \mathfrak{M}\sin\mathfrak{T}\Theta\sin\Theta)} \\ &= \sqrt{2(1 - \cos\mathfrak{T}\Theta\,\mathfrak{M}\cos\Theta - \sin\mathfrak{T}\Theta\,\mathfrak{M}\sin\Theta)}\end{aligned}$$

*) C'est le cas appelé "Cas exceptionnel" dans [1] et où la valeur typique $T\Theta$ est indéterminée.

**) Résultat donné dans [1].

et, faisant usage des relations (2),

$$\begin{aligned}\mathfrak{M}|e^{i\Theta}-e^{i\Theta}|^2 &= \sqrt{2(1-\sqrt{\mathfrak{M}^2\sin\Theta+\mathfrak{M}^2\cos\Theta)}}\\ &= \sqrt{2(1-\cos\mathfrak{D}\Theta)}\\ &= 2\sin\frac{\mathfrak{D}\Theta}{2}.\end{aligned}$$

Dispersion de l'ensemble de directions d'une courbe

Soit C une courbe rectifiable de longueur S et dont les extrémités sont aux points z_0 et Z. Si l'on suppose que la probabilité de la direction de l'élément ds est proportionnelle à ds, on aura

$$(4)\quad \cos\mathfrak{D}\Theta=\sqrt{\left(\int\frac{\sin\,ds\theta}{S}\right)^2+\left(\int\frac{\cos\,\theta\,ds}{S}\right)^2}=\sqrt{\left(\frac{X-x_0}{S}\right)^2+\left(\frac{Y-y_0}{S}\right)^2}=\frac{L}{S},$$

L étant la longueur de la droite joignant les extrémités.

Comme la direction typique des directions d'un telle courbe est la direction de la droite joignant ses extrémités *), nous avons le résultat:

La valeur typique et la dispersion de l'ensemble des directions d'une courbe rectifiable ne dépendent que de la longueur de la courbe et de la position de ses extrémités.

Directions dans un espace à trois dimensions

Soit Φ une direction aléatoire dans un espace à trois dimensions et soit $F(\varphi)$ sa loi de distribution. Nous pouvons la considérer comme un point aléatoire (L, M, N) de la sphère unitaire. La mesure de dispersion y relative est définie par la relation:

$$(5)\quad \cos\mathfrak{D}\Phi=+\sqrt{\mathfrak{M}^2L+\mathfrak{M}^2M+\mathfrak{M}^2N}.$$

Cette mesure de la dispersion a les propriétés suivantes:

1′. $\mathfrak{D}\Phi$ est un arc de grand cercle de la sphère unitaire.

2′. $\mathfrak{D}\Phi$ existe toujours quelle que soit la loi de distribution.

3′. $0\leqslant\mathfrak{D}\Phi<\frac{\pi}{2}$.

4′. $\mathfrak{D}\Phi$ est indépendant du système d'axes choisi.

5′. Quand la corde joignant les deux points φ_1 et φ_2 de la sphère est prise comme distance (φ_1,φ_2) entre ces deux points, le minimum de $\mathfrak{M}(\varphi,\Phi)^2$, valeur moyenne du carré de cette distance, est atteint quand $\varphi=\mathfrak{T}\Phi$ [*]. Le minimum de cette distance sous-tend un arc de grand cercle $\mathfrak{D}\Phi$.

Dans le cas d'une courbe rectifiable, $\cos\mathfrak{D}\Phi=L/S$, où S est la longueur de la courbe et L la longueur de la droite joignant ses extrémités.

*) Résultat donné dans [1].

Généralisation de la formule (3)

Considérons la variable x telle que

$$x = t - mp \quad \text{pour} \quad mp \leqslant t < (m+1)p \quad m \text{ entier}, \tag{6}$$

et soit X une variable aléatoire définie sur l'intervalle de $x(0 \leqslant x < p)$ avec la loi de distribution $F(x)$. Désignons alors par aX la nouvelle variable définie sur l'intervalle $(0 \leqslant x < ap)$ avec la loi de distribution $F(ax)$. En particulier, la variable $\theta = 2\pi/p\ X$ est définie sur le cercle unitaire et sa dispersion est donnée par (3). De ceci l'on tire pour $\mathfrak{D}X$, la dispersion de la variable aléatoire X, la formule

$$\cos\frac{2\pi}{p}\mathfrak{D}X = +\sqrt{\mathfrak{M}^2 \sin\frac{2\pi}{p}X + \mathfrak{M}^2\cos\frac{2}{p}X}. \tag{7}$$

La mesure de dispersion $\mathfrak{D}X$ a les propriétés suivantes:

1''. $\mathfrak{D}X$ est un segment de l'intervalle $(0 \leqslant x < p)$.

2''. $\mathfrak{D}X$ existe toujours quelle que soit la loi de distribution.

3''. $0 \leqslant \mathfrak{D}X \leqslant p/4$.

4''. $\mathfrak{D}X$ est indépendante de l'origine choisie.

5''. $\log\cos\mathfrak{D}X$ est additif quand les variables sont indépendantes et définies sur le même intervalle.

6''. $\mathfrak{D}X$ correspond au minimum de $\mathfrak{M}|e^{\frac{2\pi}{p}x} - e^{\frac{2\pi}{p}X}|^2$.

7''. $\mathfrak{D}aX = a\mathfrak{D}X$ pour $a =$ constante.

Cas particulier

Cette généralisation permet de déterminer une mesure de la dispersion d'un ensemble de directions dont le sens n'est pas défini. Ici, la variable ne peut prendre que les valeurs de l'intervalle $(0 \leqslant \theta < \pi)$ et peut être considérée comme une variable x définie sur l'intervalle $(0, \pi)$. La mesure de sa dispersion sera donc donnée par

$$\cos 2\mathfrak{D}\Theta = +\sqrt{\mathfrak{M}^2 \sin 2\Theta + \mathfrak{M}^2 \cos 2\Theta}. \tag{8}$$

UNIVERSITÉ DU CAIRE

OUVRAGES CITÉS

[1] J. Simaika, *Sur une valeur typique d'une distribution de directions*, C. R. de l'Acad. des Sci. de Paris, **241**, 1375.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 11, 1956

On a Covariant Formulation of the Equations of Motion

by

L. INFELD and J. PLEBAŃSKI

Presented by L. INFELD on August 28, 1956

1. Introduction. There are two ways of approaching the problem of motion in general relativity theory: by deducing the equations of motion from the field equations [1]-[5], and by using the principle of the "geodetic line" [6], [7] (the meaning of the quotation marks will become clear later). It is known that the two approaches give the same concrete results up to the post-Newtonian order. But, we believe, it has never been shown that these two approaches must always give precisely the same equations of motion under certain conditions, specified in this paper.

2. The field equations. We assume the following equations for the field of singularities:

$$G^{\alpha\beta} = -8\pi \sum_{A=1}^{N} \overset{A}{\mu}(x^0)\hat{\delta}(\vec{x} - \vec{\overset{A}{\xi}}(x^0))\overset{A}{\xi}{}^{\alpha}(x^0)_{|0}\overset{A}{\xi}{}^{\beta}(x^0)_{|0}. \tag{2.1}$$

Here $G^{\alpha\beta}$ is the density of Einstein's tensor $G^{\alpha\beta} = \sqrt{-g}(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R)$. The symbols $\overset{A}{\xi}{}^{\alpha} = \overset{A}{\xi}{}^{\alpha}(x^0)$, $\overset{A}{\xi}{}^{0} = x^0$ denote the world lines of the $N (A = 1, 2, ..., N)$ particles. "Time" — $x^0 = ct$ — is regarded as the parameter. Stroke means ordinary differentiation: $S(x^\mu)_{|\alpha} = \frac{\partial}{\partial x^\alpha} S(x^\mu)$.

By $\hat{\delta}(\vec{x})$ we denote the "good" three dimensional Dirac δ-function [8], which besides satisfying the condition:

$$\int \hat{\delta}(\vec{x} - \vec{\xi}) f(\vec{x})\, d_3x = f(\vec{\xi})$$

for continuous functions, also satisfies the equations:

$$\int \frac{\hat{\delta}(\vec{x})}{|\vec{x}|^p}\, d_3x = 0, \qquad p = 1, 2, ..., k. \tag{2.2}$$

Since the integral $\frac{dx^\circ}{ds_A}\int\limits_{\overset{A}{\Omega}} G^{\alpha\beta}d_3x$ taken over a small region surrounding the A'th singularity $\vec{x}=\overset{A}{\vec{\xi}}(x^\circ)$ is a tensor, therefore $\overset{A}{\mu}(x^\circ)=\overset{A}{\mu_0}\frac{dx^\circ}{ds_A}$, where $\overset{A}{\mu_0}$ is an invariant, and ds_A the line element of the A-th particle *).

Going back to (2.1) we conclude that $\hat{\delta}$ has the transformation properties of the zero'th component of a tensor density.

3. The equations of motion and the field equations. We know that because of the Bianchi identities we have (where a semicolon denotes covariant differentiation):

$$G^{\alpha\beta}{}_{;\beta}=0. \tag{3.1}$$

Therefore we must also have:

$$\sum_{A=1}^{N}\Big(\overset{A}{\mu}(x^\circ)\hat{\delta}(\vec{x}-\overset{A}{\vec{\xi}}(x^\circ))\overset{A}{\xi}{}^{\alpha}(x^\circ)_{|0}\overset{A}{\xi}{}^{\beta}(x^\circ)_{|0}\Big)_{;\beta}=0. \tag{3.2}$$

Written out explicitly, since both sides are tensor densities, this gives:

$$\sum_{A=1}^{N}\Big[\big(\overset{A}{\mu}\hat{\delta}\overset{A}{\xi}{}^{\beta}_{|0}\overset{A}{\xi}{}^{\alpha}_{|0}\big)_{|\beta}+\left\{\begin{matrix}\alpha\\ \nu\mu\end{matrix}\right\}\overset{A}{\mu}\hat{\delta}\overset{A}{\xi}{}^{\nu}_{|0}\overset{A}{\xi}{}^{\mu}_{|0}\Big]=0. \tag{3.3}$$

We shall now introduce the following notation

$$\overset{A}{\tilde{f}}=\int d_3x\hat{\delta}(\vec{x}-\overset{A}{\vec{\xi}}(x^\circ))f(\vec{x},x^\circ). \tag{3.4}$$

The operation "$\underset{\sim}{A}$" does two things. Firstly, because of (2.2) it selects only the regular part of f, suppressing the singularities. Secondly, it introduces the ξ^{α}'s instead of the x^{α}'s. Thus we shall always use

$$f(x^\circ,\vec{x})\hat{\delta}(\vec{x}-\overset{A}{\vec{\xi}}(x^\circ))=\overset{A}{\tilde{f}}\hat{\delta}(\vec{x}-\overset{A}{\vec{\xi}}(x^\circ)). \tag{3.5}$$

Therefore and because of (3.5) we can write (3.3):

$$\sum_{A=1}^{N}\Big[\big(\overset{A}{\mu}\hat{\delta}(\vec{x}-\overset{A}{\vec{\xi}}(x^\circ))\overset{A}{\xi}{}^{\beta}_{|0}\overset{A}{\xi}{}^{\alpha}_{|0}\big)_{|\beta}+\overset{A}{\mu}\left\{\begin{matrix}\overset{A}{\alpha}\\ \nu\mu\end{matrix}\right\}\overset{A}{\xi}{}^{\nu}_{|0}\overset{A}{\xi}{}^{\mu}_{|0}\hat{\delta}(\vec{x}-\overset{A}{\vec{\xi}}(x^\circ))\Big]=0. \tag{3.6}$$

We see that:

*) More about this in § 4. Here $\overset{A}{\mu_0}$ is the rest mass; in ordinary units $\overset{A}{\mu_0}=\frac{k\overset{A}{m_0}}{c^2}$, k being the gravitational constant and c the velocity of light.

$$(3.7)\qquad \left(\overset{A}{\mu}\hat{\delta}(\vec{x}-\overset{A}{\vec{\xi}}(x^\circ))\overset{A}{\xi}{}^{\beta}_{|0}\overset{A}{\xi}{}^{\alpha}_{|0}\right)_{|\beta}=(\overset{A}{\mu}\hat{\delta}\overset{A}{\xi}{}^{\alpha}_{|0})_{|0}+\overset{A}{\mu}\hat{\delta}_{|\sigma}\overset{A}{\xi}{}^{\sigma}_{|0}\overset{A}{\xi}{}^{\alpha}_{|0}$$

$$=(\overset{AA}{\mu\xi}{}^{\sigma}_{|0})_{|0}\hat{\delta}(\vec{x}-\overset{A}{\vec{\xi}}(x^\circ)).$$

Making use of (3.7) in (3.6) we have:

$$(3.8)\qquad \sum_{A=1}^{N}\hat{\delta}(\vec{x}-\overset{A}{\vec{\xi}}(x^\circ))\left[(\overset{AA}{\mu\xi}{}^{\alpha}_{|0})_{|0}+\overset{A}{\mu}\left\{\begin{matrix}\overset{A}{\alpha}\\ \nu\mu\end{matrix}\right\}\overset{A}{\xi}{}^{\nu}_{|0}\overset{A}{\xi}{}^{\mu}_{|0}\right]=0.$$

Thus the necessary and sufficient condition for (3.8) are equations (3.9):

$$(3.9)\qquad (\overset{A}{\mu}(x^\circ)\overset{A}{\xi}{}^{\alpha}_{|0})_{|0}+\overset{A}{\mu}(x^\circ)\left\{\begin{matrix}\overset{A}{\alpha}\\ \nu\mu\end{matrix}\right\}\overset{A}{\xi}{}^{\nu}_{|0}\overset{A}{\xi}{}^{\mu}_{|0}=0;\qquad \overset{A}{\xi}{}^{\circ}=x^\circ.$$

Indeed, if the equations of (3.9) are fulfilled those of (3.8) are satisfied. On the other hand, if (3.8) is satisfied we find (3.9) by integrating (3.8) over $\overset{A}{\Omega}$.

Thus the equations of (3.9) are the equations of motion; they must be satisfied by all world lines $\overset{A}{\xi}{}^{\alpha}(x^\circ)$. They follow as a consequence of the field equations (2.1); if developed into a power series in $1/c$ they give us the Newtonian and post-Newtonian equations of motion.

4. The *s*-formalism. So far in our considerations we have treated the variable $x^\circ = ct$ in a special way. We shall say, that we used the "t-formalism". But all that has been done can also be done in a covariant form, using the line element ds_A. Hence the name "s-formalism".

Let us assume, as is true in all special cases so far investigated, that the field $g_{\alpha\beta}(x^\nu)$ has the following properties:

$$(4.1)\qquad \overset{A}{g^{\alpha\varrho}}g_{\varrho\beta}=\overset{A}{g^{\alpha\varrho}}\overset{A}{g}_{\varrho\beta},$$

$$(4.2)\qquad \left\{\begin{matrix}\overset{A}{\alpha}\\ \nu\mu\end{matrix}\right\}\equiv\overset{A}{g^{\alpha\varrho}}[\nu\mu,\varrho]=\overset{A}{g}{}^{\alpha\varrho}\overset{A}{[\nu\mu,\varrho]}.$$

These would be trivial properties for continuous fields. They are not trivial for fields with singularities.

We shall now show, using (4.1) and (4.2), that (3.9) represents the equation of "geodetic lines". In equations (3.9) we have $4N$ equations for $4N$ unknown functions: $\overset{A}{\xi}{}^{\alpha}(x^\circ)$ and $\overset{A}{\mu}(x^\circ)$, $A=1,2,\ldots,N$. Let us now introduce the line element ds_A mentioned in **2** in place of $\overset{A}{\mu}$.

We therefore define:

$$(4.3)\qquad ds_A \overset{\mathrm{df}}{=} \frac{\overset{A}{k}m_0}{c^2}\frac{dx^\circ}{\overset{A}{\mu}(x^\circ)}.$$

This gives us s_A as a function of x°_A, or, x° as a function of s_A. Therefore, if we introduce x° as a function of s_A into $\left\{\begin{matrix}\overset{A}{\sim}\\ \alpha\\ \nu\mu\end{matrix}\right\}$ we can regard this function as a function of s_A. Let us, therefore rewrite (3.9) using s_A as a variable. We have:

$$\frac{ds_A}{dx^\circ}\frac{d}{ds_A}\left(\overset{A}{\mu}\frac{ds^A}{dx^\circ}\frac{d}{ds_A}\overset{A}{\xi^\alpha}\right)+\overset{A}{\mu}\left(\frac{ds_A}{dx^\circ}\right)^2\left\{\begin{matrix}\overset{A}{\sim}\\ \alpha\\ \nu\mu\end{matrix}\right\}\frac{d\overset{A}{\xi^\nu}}{ds_A}\frac{d\overset{A}{\xi^\mu}}{ds_A}=0. \tag{4.4}$$

Using (4.3) to eliminate $\overset{A}{\mu}$ we have

$$\frac{\overset{A}{km_0}}{c^2}\frac{ds_A}{dx^\circ}\left[\frac{d^2}{ds_A^2}\overset{A}{\xi^\alpha}+\left\{\begin{matrix}\overset{A}{\sim}\\ \alpha\\ \nu\mu\end{matrix}\right\}\frac{d\overset{A}{\xi^\nu}}{ds_A}\frac{d\overset{A}{\xi^\mu}}{ds_A}\right]=0 \tag{4.5}$$

or

$$\overset{A}{Q^\alpha}(s_A)\equiv\frac{d^2}{ds_A^2}\overset{A}{\xi^\alpha}+\left\{\begin{matrix}\overset{A}{\sim}\\ \alpha\\ \nu\mu\end{matrix}\right\}\frac{d\overset{A}{\xi^\nu}}{ds_A}\frac{d\overset{A}{\xi^\mu}}{ds_A}=0. \tag{4.6}$$

These are the equations which were called throughout this paper the „geodetic equations", because of their external form. Indeed we can show that the ds_A used here and defined by (4.3) are the "line element" (up to a constant), that is

$$Q\equiv\overset{A}{\underset{\sim}{g}}_{\alpha\beta}\frac{d\overset{A}{\xi^\alpha}}{ds_A}\frac{d\overset{A}{\xi^\beta}}{ds_A}=\text{const.} \tag{4.7}$$

To prove this we must differentiate Q with respect to ds_A. What do we understand by $\frac{d}{ds_A}\overset{A}{\underset{\sim}{g}}_{\alpha\beta}$?. The answer is obvious: Since $g_{\alpha\beta}$, a concrete solution of the field equations depends on $\vec{x},x^\circ,\overset{1}{\xi^\alpha}(x^\circ),\ldots\overset{N}{\xi^\alpha}(x^\circ)$, $\overset{1}{\xi^\alpha}(x^\circ)_{|0},\ldots$ we have

$$g_{\nu\beta|0}=\frac{\partial g_{\alpha\beta}}{\partial(x^\circ)}+\sum_{\overset{B}{B}}\frac{\partial g_{\alpha\beta}}{\partial\overset{B}{\xi^a}}\overset{B}{\xi^a_{|0}}+\ldots \tag{4.8}$$

and:

$$\frac{d}{ds_A}\overset{A}{\underset{\sim}{g}}_{\alpha\beta}=\frac{d}{ds_A}\int d_3x\,\hat{\delta}(\vec{x}-\overset{A}{\vec{\xi}}(x^\circ))g_{\alpha\beta}(\vec{x},x^\circ,\ldots)=$$

$$=\int d_3x\left(-\hat{\delta}_{|r}\cdot\frac{d\overset{A}{\xi^\nu}}{ds_A}\cdot g_{\alpha\beta}+\hat{\delta}\cdot g_{\alpha\beta|0}\frac{dx^\circ}{ds_A}\right).$$

Therefore:

$$\frac{d}{ds_A}\overset{A}{\underset{\sim}{g}}_{\alpha\beta}=\overset{A}{\underset{\sim}{g}}_{\alpha\beta|r}\frac{d\overset{A}{\xi^r}}{ds_A}+\overset{A}{\underset{\sim}{g}}_{\alpha\beta|0}\frac{d\overset{A}{\xi^\circ}}{ds_A}=\frac{d\overset{A}{\xi^\varrho}}{ds_A}\overset{A}{\underset{\sim}{g}}_{\alpha\beta|\varrho}. \tag{4.9}$$

From (4.9), introducing (4.6) into the results of differentiation, we have:

$$\frac{d}{ds_A}Q=\frac{d\overset{A}{\tilde{g}}_{\alpha\beta}}{ds_A}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}+2\overset{A}{\tilde{g}}_{\alpha\beta}\frac{d^2\overset{A}{\xi}{}^{\alpha}}{ds_A^2}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}= \tag{4.10}$$

$$=\overset{A}{\tilde{g}}_{\alpha\beta|\varrho}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}\frac{d\overset{A}{\xi}{}^{\varrho}}{ds_A}-2\overset{A}{\tilde{g}}_{\alpha\beta}\left\{\begin{matrix}\alpha\\ \nu\mu\end{matrix}\right\}\frac{d\overset{A}{\xi}{}^{\nu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\mu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}.$$

Thus, because of $g^{\alpha\varrho}\,g_{\varrho\beta}=\delta^{\alpha}_{\beta}$ and because of (4.1), (4.2) we can rewrite:

$$\frac{dQ}{ds_A}=\overset{A}{\tilde{g}}_{\alpha\beta|\varrho}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}\frac{d\overset{A}{\xi}{}^{\varrho}}{ds_A}-(\overset{A}{\tilde{g}}_{\nu\beta|\mu}+\overset{A}{\tilde{g}}_{\mu\beta|\nu}-\overset{A}{\tilde{g}}_{\nu\mu|\beta})\frac{d\overset{A}{\xi}{}^{\nu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\mu}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}=0 \tag{4.11}$$

after trivial changes of indices. Therefore:

$$Q=\overset{A}{\tilde{g}}_{\alpha\beta}\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}=\text{const.} \tag{4.12}$$

Thus we have shown that from the field equations in which particles are regarded as singularities of the field, the equations of motion of a "geodetic line" follow.

It only remains to show that the right hand side of the field equations can also be rewritten in a covariant form by the use of the s-formalism.

We define $\hat{\delta}_4(x^{\mu})$ by

$$\int_{-\infty}^{+\infty}d\lambda\,\delta_4(x^{\mu}-\xi^{\mu}(\lambda))=\frac{d\lambda}{d\xi^{\circ}}\Big|_{\xi^{\circ}=x^{\circ}}\hat{\delta}(\vec{x}-\vec{\xi}(x^{\circ})). \tag{4.13}$$

Then we can rewrite the field equations in the form

$$G^{\alpha\beta}=-\frac{8\pi k}{c^2}\sum_{A=1}^{N}\overset{A}{m}_0\int_{-\infty}^{+\infty}ds_A\hat{\delta}_4(x^{\mu}-\overset{A}{\xi}{}^{\mu}(s_A))\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}, \tag{4.14}$$

where

$$ds_A=(\overset{A}{\tilde{g}}_{\alpha\mu}d\overset{A}{\xi}{}^{\alpha}d\overset{A}{\xi}{}^{\beta})^{1/2}. \tag{4.15}$$

These field equations as well as the equations of motion in the form (4.6) can be gained from a variational principle:

$$\delta\Big(\sum_{A=1}^{N}-\overset{A}{m}_0c\int(\overset{A}{\tilde{g}}_{\alpha\beta}d\overset{A}{\xi}{}^{\alpha}d\overset{A}{\xi}{}^{\beta})^{1/2}+\frac{c^3}{16\pi k}\int d_4x\sqrt{-g}R\Big)=0, \tag{4.16}$$

where the variation is performed with respect to the $\overset{A}{\xi}{}^{\alpha}(s_A)$ and $g_{\alpha\beta}(x^{\mu})$.

This and other considerations concerning the problem of motion will be given more fully in a monograph prepared by the authors of this note.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
INSTITUTE OF THEORETICAL PHYSICS OF WARSAW UNIVERSITY

REFERENCES

[1] A. Einstein and J. Grommer, Sitz. Berl. Akad. Wiss. **1** (1927).
[2] A. Einstein, L. Infeld and B. Hoffman, Ann. Math. **39** (1938), 66.
[3] A. Einstein and L. Infeld, Ann. Math. **41** (1940), 455.
[4] — Can. J. Math. **1** (1949), 209.
[5] L. Infeld, Acta Phys. Pol. **13** (1954), 187.
[6] A. S. Eddington, G. L. Clark, Proc. Roy. Soc. A, **166** (1938), 465.
[7] B. Bertotti, Nuovo Cimento **12** (1954), 226.
[8] L. Infeld and J. Plebański, *On modified Dirac δ-functions*, Bull. Acad. Polon. Sci. Cl. III, **4** (1956), 689.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 11, 1956

THEORETICAL PHYSIC

On the "Dipole Procedure" in General Relativity Theory

by

L. INFELD and J. PLEBAŃSKI

Presented by L. INFELD on August 28, 1956

1. Introduction. Let us consider the equations of general relativity theory *)

$$G^{\alpha\beta} = -\frac{8\pi k}{c^2}\sum_{A=1}^{N} \overset{A}{m}_0 \int_{-\infty}^{+\infty} ds_A \hat{\delta}_4\left(x^\mu - \overset{A}{\xi}{}^\mu(s_A)\right) \frac{d\overset{A}{\xi}{}^\alpha}{ds_A}\frac{d\overset{A}{\xi}{}^\beta}{ds_A} \tag{1.1}$$

$$\overset{A}{\Omega}{}^\alpha(s_A) = \frac{d^2}{ds_A}\overset{A}{\xi}{}^\alpha + \left\{\begin{matrix}\overset{A}{\alpha}\\ \nu\mu\end{matrix}\right\}\frac{d\overset{A}{\xi}{}^\nu}{ds_A}\frac{d\overset{A}{\xi}{}^\mu}{ds_A} = 0 \tag{1.2}$$

$$ds_A = (\overset{A}{g}_{\alpha\beta} d\overset{A}{\xi}{}^\alpha d\overset{A}{\xi}{}^\beta)^{1/2}. \tag{1.3}$$

Let us assume that we wish to find the motion of the particles, that is the explicit values for $\overset{A}{\xi}{}^\alpha(s_A)$, $A = 1, 2, \ldots N$. To do that we would have to know the field since the motion is determined by the field equations. But to find the field we would have to know the motion, since the field is determined by motion, that is world lines on the right hand side of (1.1). This difficulty was met very early [3] in trying to formulate the general theory of solving (1.1). The idea used then was the following one. Gene-

*) We repeat here briefly the notation given in a previous paper [1]. $G^{\alpha\beta} = \sqrt{-g}(R^{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R)$ is the Einstein's tensor density. $\overset{A}{\xi}{}^\alpha(s_A)$ are the world lines of N particles $(A = 1, 2, \ldots N)$. $\hat{\delta}_4$ is defined by $\int_{-\infty}^{+\infty} d\lambda \hat{\delta}_4\left(x^\mu - \xi^\mu(\lambda)\right) = \frac{d\lambda}{\delta\xi^0}\hat{\delta}\left(\vec{x} - \vec{\xi}(\xi^0)\right)_{|\xi^0 = x^0}$; $\hat{\delta}(\vec{x})$ is the three dimensional "good" Dirac δ-function (introduced in [2]) which satisfies $\int d_3x \hat{\delta}(\vec{x}) f(\vec{x}) = f(0)$ where f is a continuous function and satisfies the conditions: $\int d_3x \frac{\hat{\delta}(\vec{x})}{|\vec{x}|^p} = 0$, $p = 1, 2, \ldots k$. Operation "$\underset{\sim}{A}$" is defined as

$$\overset{A}{\widetilde{f}} = \int \hat{\delta}\left(\vec{x} - \overset{A}{\vec{\xi}}(x^0)\right) f(\vec{x}, x^0)\, d_3x.$$

ralise (1.1) in such a way that the motion is not determined by field equations. That is, instead of solving (1.1) solve equations different from (1.1), non Einstein equations, in which the motion becomes arbitrary by adding certain dipole fields to the right hand side of (1.1). Then go back to Einstein's equations by assuming that the additional field vanishes. This procedure has two shortcommings 1° It is not a covariant procedure, 2° It is closely connected with the approximation procedure used in solving (1.1). We have now found that the two shortcomings can be removed, as we shall show in this note.

2. The mathematical formulation of the problem. We know from [1] that by using Bianchi's identities $G^{\alpha\beta}{}_{;\beta}=0$ on (1.1) we can deduce (1.2). Equations (1.2) are the equations of motion forming a condition of consistency for equations (1.1).

We wish to add such new fields to (1.1) in a covariant way, that the equations of motion will be wiped out; so that the motion will become arbitrary.

We now introduce a "dipole field" D^{α} and we take as our new set of equations the following

$$(2.1)\qquad G^{\alpha\beta}=-\frac{8\pi k}{c^2}\sum_{A=1}^{N}\overset{A}{m}_0\int_{-\infty}^{+\infty}ds_A\hat{\delta}_4(x^{\mu}-\overset{A}{\xi}{}^{\mu}(s_A))\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}+D^{\alpha;\beta}+D^{\beta;\alpha}-g^{\alpha\beta}D^{\varrho}{}_{;\varrho},$$

$$(2.2)\qquad D^{\alpha;\beta}{}_{;\beta}+R^{\alpha\beta}D_{\beta}=\frac{8\pi k}{c^2}\sum_{A=1}^{N}\overset{A}{m}_0\int_{-\infty}^{+\infty}ds_A\overset{A}{\delta}_4(x^{\mu}-\overset{A}{\xi}{}^{\mu}(s_A))\overset{A}{\Omega}{}^{\alpha}(s_A).$$

First: Let us show that the motion is arbitrary. Indeed, using the new Bianchi identities in (2.1), we obtain, because of (2.2)

$$(2.3)\qquad \left[\frac{-8\pi k}{c^2}\sum_{A=1}^{N}\overset{A}{m}_0\int_{-\infty}^{+\infty}ds_A\hat{\delta}_4(x^{\mu}-\overset{A}{\xi}{}^{\mu})\frac{d\overset{A}{\xi}{}^{\alpha}}{ds_A}\frac{d\overset{A}{\xi}{}^{\beta}}{ds_A}+D^{\alpha;\beta}+D^{\beta;\alpha}-g^{\alpha\beta}D^{\varrho}{}_{;\varrho}\right]_{;\beta}\equiv 0,$$

because we can show that equations (2.3) and (2.2) are identical in form and content (compare [1]).

Thus by using Bianchi's identities on the right side of equation (2.1) we get identically zero, and no conditions are forced upon the motion of the particles. Therefore the solutions of (2.1) and (2.2) exist for arbitrary motion.

Second: Let us take $D^{\alpha}\equiv 0$. Then our equations (2.1) and (2.2) go over into (1.1) (that is into Einstein's equations) and (1.2), i. e. the equations of motion. The vanishing of the right hand side of (2.2) is equivalent with (1.2).

Third: If $\overset{A}{\Omega}{}^{\alpha}=0$, then it follows from (3.2) that D^{α} is not necessarily equal to zero. We shall, however, assume that we always take only the solution of non-homogeneous equations, that is $\overset{A}{\Omega}{}^{\alpha}(s_A)=0 \rightarrow D^{\alpha}\equiv 0$.

Since the motion is arbitrary, we can just as well assume $\overset{A}{\Omega}{}^{\alpha}(s_A)=0$. What do we achieve by this assumption? The $g_{\alpha\beta}$ as functions of x^{α} and functionals of $\overset{A}{\xi}{}^{\alpha}$ are now uniquely determined. The field equations (2.2) give us $D^{\alpha}\equiv 0$.

The generalised field equations (2.1) become Einstein equations. Thus we have solved our original problem by introducing the auxiliary field D^{α}.

3. The dipole procedure and the method of approximation. The approximation procedure will make it clear why we call the D^{α} field, the "dipole field".

We shall rewrite our equations (2.1), (2.2) using the t instead of the s formalism as:

$$(3.1)\qquad G^{\alpha\beta}=-8\pi\sum_{A=1}^{N}\overset{A}{\mu}(x^{\circ})\,\hat{\delta}\big(\vec{x}-\vec{\overset{A}{\xi}}(x^{\circ})\big)\,\overset{A}{\xi}{}^{\alpha}(x^{\circ})_{|0}\,\overset{A}{\xi}{}^{\beta}(x^{\circ})_{|0}+D^{\alpha;\beta}+D^{\beta;\alpha}-g^{\alpha\beta}D^{\varrho}{}_{;\varrho},$$

$$(3.2)\qquad D^{\alpha;\beta}{}_{;\beta}+R^{\alpha\beta}D_{\beta}=8\pi\sum_{A=1}^{N}\hat{\delta}\big(\vec{x}-\vec{\overset{A}{\xi}}(x^{\circ})\big)\Big[\big(\overset{A}{\mu}(x^{\circ})\,\overset{A}{\xi}{}^{\alpha}{}_{|0}\big)_{|0}+\overset{A}{\mu}(x^{\sigma})\left\{\begin{matrix}\alpha\\ \nu\mu\end{matrix}\right\}\overset{A}{\xi}{}^{\nu}{}_{|0}\,\overset{A}{\xi}{}^{\mu}{}_{|0}\Big],$$

where:

$$(3.3)\qquad \overset{A}{\mu}(x^{\circ})=\frac{k\overset{A}{m}_0}{c^2}\frac{dx^{\circ}}{ds_A}.$$

As far as the $g_{\alpha\beta}$ and μ are concerned we assume development in $1/c$ and write its order underneath [3], [4].

$$(3.4)\qquad \begin{aligned} g_{oo}&=1+\underset{2}{h_{oo}}+\underset{4}{h_{oo}}+\underset{6}{h_{oo}}+\dots\\ g_{nm}&=-\delta_{nm}+\underset{2}{h_{nm}}+\underset{4}{h_{nm}}+\underset{6}{h_{nm}}+\dots\\ g_{on}&=\underset{3}{h_{on}}+\underset{5}{h_{on}}+\underset{7}{h_{on}}+\dots\\ \overset{A}{\mu}&=\underset{2}{\overset{A}{\mu}}+\underset{4}{\overset{A}{\mu}}+\underset{6}{\overset{A}{\mu}}+\dots \end{aligned}$$

(consistently with $\underset{2}{\overset{A}{\mu}}=\overset{A}{km}_0/c^2$).

Because (as we assume in our approximation procedure) the derivative with respect to $x^{\circ}=ct$ is of order 1 higher than with respect to x^{a} $(a=1,2,3)$, the right hand side of (3.2) for $\alpha=0$ begins with order 3 and for $\alpha=a$ with order 4.

Therefore we have the developments for $\underset{a}{D}$:

$$\begin{aligned} \underset{}{D}^{\circ} &= \underset{3}{D}^{\circ} + \underset{5}{D}^{\circ} + \underset{7}{D}^{\circ} + \dots \\ \underset{a}{D} &= \underset{4}{D}_{a} + \underset{6}{D}_{a} + \underset{8}{D}_{a} + \dots \end{aligned} \tag{3.5}$$

Thus we can easily see that for the beginning of our approximation procedure in equations (3.2) we have:

$$\begin{aligned} \underset{3}{D}^{\circ}{}_{|ss} &= 8\pi \sum_{A=1}^{N} \underset{2}{\overset{A}{\mu}}{}_{|0} \underset{1}{\overset{A}{\delta}}(\vec{x} - \vec{\xi}(x^{\circ})), \\ \underset{4}{D}^{a}{}_{|ss} &= 8\pi \sum_{A=1}^{N} \left[\underset{2}{\overset{A}{\mu}} \underset{1}{\overset{A}{\xi}}{}^{a}{}_{|0|0} + \underset{2}{\overset{A}{\mu}} \left\{ \begin{matrix} \tilde{a} \\ 00 \end{matrix} \right\}_{2}^{A} \right] \hat{\delta}(\vec{x} - \vec{\xi}(x^{\circ})). \end{aligned} \tag{3.6}$$

Therefore in agreement with our condition that the $\underset{a}{D}$'s have to vanish if the sources of equation (3.2) vanish we have:

$$\underset{3}{D}^{\circ} = -\sum_{A=1}^{N} 2 \underset{2}{\overset{A}{\mu}}{}_{|0} \frac{1}{|\vec{x} - \vec{\xi}(x^{\circ})|}, \qquad \underset{4}{D}^{a} = -\sum_{A=1}^{N} 2 \left[(\underset{2}{\overset{AA}{\mu}} \underset{1}{\overset{A}{\xi}}{}^{a}{}_{|0})_{|0} + \underset{2}{\overset{A}{\mu}} \left\{ \begin{matrix} \tilde{a} \\ 00 \end{matrix} \right\}_{2}^{A} \right] \cdot \frac{1}{|\vec{x} - \vec{\xi}(x^{\circ})|}. \tag{3.7}$$

This put into (3.1) gives us the "dipole fields"; terms of the form:

$$f^{s}(x^{\circ}) \frac{\partial}{\partial x^{s}} \frac{1}{|\vec{x} - \vec{\xi}(x^{\circ})|}.$$

But the dipole fields will appear in (3.1) only in the 3-th and 4-th orders, therefore in the post-Newtonian order only. Therefore the equations for the Newtonian field can be solved for an arbitrary motion.

The dipole tensor added in the post-Newtonian order, is formed exclusively from expressions known from the solution in the Newtonian order. Yet if we want the equations for the field, which we can now solve for an arbitrary motion, to go over into the right Einstein equations, we must assume

$$\underset{3}{D}^{\circ} = 0, \qquad \underset{4}{D}^{\circ} = 0. \tag{3.8}$$

But (3.8) are valid if, and only if, the Newtonian equations are fulfilled and $\underset{2}{\overset{A}{\mu}}$ is constant.

Of course, this situation will repeat itself in the higher orders. We shall get the post-Newtonian equations of motion as conditions for

$$\underset{3}{D}^{\circ} + \underset{5}{D}^{\circ} = 0, \qquad \underset{4}{D}^{a} + \underset{6}{D}^{a} = 0, \tag{3.9}$$

which is a condition for returning to the Einstein equations for the post-post-Newtonian field.

This procedure, throwing new and interesting light upon the general theory, will be of little practical value if we wish to obtain the equations of motion.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
INSTITUTE OF THEORETICAL PHYSICS, WARSAW UNIVERSITY

REFERENCES

[1] L. Infeld, J. Plebański, *On modified Dirac δ-functions*, Bull. Acad. Polon. Sci., Cl. III. **4** 1956), 689.
[2] — *On a covariant formulation of the equations of motion,* in this issue p. 757,
[3] A. Einstein, L. Infeld, Can. J. Math. **1** (1949), 209.
[4] L. Infeld, Acta Phys. Pol., **13** (1954), 187.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 11, 1956

PHYSIQUE EXPÉRIMENTALE

Sur l'influence de la température sur la largeur des raies de tetrachlorure de carbone

par

B. MOSZYŃSKA

Présenté par A. JABŁOŃSKI le 10 Octobre 1956

Introduction

Le problème de la largeur des raies ramaniennes, surtout pour la substance en état liquide, est encore un des problèmes le moins élaboré dans le domaine des spectres ramaniens.

Conformément aux considérations de Sobielman [1] le mouvement chaotique rotatoire effectué par chaque molécule dans le liquide ayant pour conséquence les changements d'orientation des molécules les unes envers les autres, est une des causes principales de l'élargissement de la raie. Cet élargissement augmente avec l'accroissement de l'anisotropie du tenseur des dérivées de la polarisabilité par rapport aux coordonnées normales, c'est-à-dire avec la dépolarisation de la raie; pour la raie totalement polarisée ($\varrho = 0$), l'élargissement ne paraît pas; pour la raie entièrement dépolarisée ($\varrho = \frac{6}{7}$) l'élargissement atteint le maximum, notamment $1/\tau$, où τ désigne le temps d'un changement d'orientation d'une molécule dans le liquide.

Afin de suivre l'influence de ce facteur sur la largeur de la raie il paraît raisonnable d'examiner les changements du spectre de la même substance, qui paraissent avec le changement de quelque paramètre, qui modifie l'influence de l'entourage moléculaire sur la molécule dispersante. Dans le travail de Bajoulin et Rakov [2], où l'on compare les spectres de la substance en état liquide avec ceux de la même substance en état solide, c'est l'état de la substance qui joue le rôle de ce paramètre; dans le travail présent c'est la température du liquide.

On a examiné le spectre du tetrachlorure de carbone aux températures de 20°, 55° et 65°C. Le spectre de CCl_4 contient quatre raies; l'une d'entre elles, ν_1, est entièrement polarisée, les trois autres sont totalement dépolarisées. Si le mouvement rotatoire brownien mentionné ci-dessus était la seule cause de cet élargissement dû à la température, la largeur de la raie ν_1 devrait rester invariable aux différentes températures et les autres raies devraient s'élargir au même degré.

Appareillage

Les spectres ramaniens ont été obtenus à l'aide du spectrographe à deux prismes "Huet B II", la dispersion duquel put être augmentée par le déplacement symétrique des prismes de la position, de la plus petite déviation [3]. Au lieu de 18 Å par millimètre sur la plaque, sa dispersion est devenue 7 Å par millimètre vers 4358 Å.

Le spectre a été excité par la raie 4358 Å de mercure provenant de la lampe en verre de basse pression du type Sosiński [4] construite dans notre laboratoire.

Résultats et conclusions

On n'a examiné que les trois plus fortes raies de CCl_4, à savoir:

1) la raie $\nu_1 = 459$ cm^{-1} correspondant à l'oscillation de valence C—Cl totalement symétrique;

2) les raies $\nu_2 = 217$ cm^{-1} et $\nu_4 = 314$ cm^{-1} correspondant à l'oscillation de déformation, doublement et triplement dégénérée.

Nous avons d'abord transformé la courbe microphotométrique des raies en celle d'intensité en utilisant les procédés habituels de photométrie photographique. Afin d'évaluer la largeur de la raie ramanienne nous avons soustrait la largeur expérimentale de la raie excitatrice, mesurée sur le profil à la demi intensité, de la largeur de la raie ramanienne expérimentale mesurée de la même façon.

Nous avons constaté que la largeur expérimentale de la raie excitatrice était pour la plupart des cas de 1,4-1,7 cm^{-1}. (La largeur effective est de 0,2 cm^{-1} environ).

Les largeurs des trois raies ramaniennes obtenues à la température de 20°C. sont données dans le tableau I à côté des résultats obtenus par

TABLEAU I

Largeurs des raies de CCl_4 à la température de 20°C

ν cm^{-1}		$\Delta\nu$ cm^{-1}	
	Souchtchinski [5]	Sterin [6]	Auteur
459	6.5	6	6.5
217	7	6.5	6.7
314	6	6	7.4

d'autres auteurs, probablement à une température analogue. On y voit que les largeurs des raies ν_1 et ν_2 sont identiques, tandis que la largeur de la raie ν_4 obtenue dans nos mesurements est plus grande.

Le fait que les largeurs de toutes les raies sont relativement grandes déjà à la température de 20°C s'explique en premier lieu par la structure isotopique des raies dûe à l'existence de deux isotopes du chlore, Cl^{35}

et Cl^{37}, dans les molécules de CCl_4. Chaque raie constitue donc une résultante de plusieurs composantes; le cas de la raie due à l'oscillation totalement symétrique est représenté à la Figure 1.

Les intervalles entre les composantes isotopiques sont deux fois plus grandes pour la raie ν_1 que pour la raie ν_2. Les largeurs des résultantes sont pourtant les mêmes; cela prouve que les composantes isotopiques de la raie ν_2 sont plus larges, conformément à la théorie qui stipule la dépendance de la largeur des raies de l'anisotropie de l'oscillation. On peut appliquer le même raisonnement à la raie ν_4 pour laquelle les intervalles isotopiques sont encore plus petits, bien que la structure de cette raie soit beaucoup plus compliquée à cause de la disparition de la dégénération pour les composantes correspondant à toutes les molécules contenant des isotopes différents.

Fig. 1

Les résultats des mesures obtenus aux températures de 20°, 55° et 65°C sont rassemblés au tableau II. Pour faciliter la comparaison on a admis que la largeur de la raie à la température de 60°C est égale à la moyenne arithmétique des largeurs des raies aux températures de 55° et 65°C. On y voit que, conformément à ce qui était prévu, les deux raies correspondant aux oscillations de déformation s'élargissent distinctement; la largeur de la raie ν_2 change plus que la largeur de la raie ν_4; de plus, "le maximum" de la raie ν_2 à la température de 60°C s'élargit. Sur quelques plaques il y a même un fendage en deux maxima éloignés l'un de l'autre de 1,7 cm^{-1} (v. Fig. 2).

D'ailleurs, contrairement aux prévisions, la raie ν_1 s'élargit aussi quoique moins que les autres. On peut expliquer cet élargissement en partie par la modification de la raie visible à la Figure 2. Notamment, l'intensité des ailes augmente avec la température ce que l'on peut observer le mieux du côté des basses fréquences de cette raie.

Cette modification du profil révèle l'apparition des bandes de différence $\nu_1+\nu_4-\nu_4$. On sait que l'état $\nu_1+\nu_4$, dont la fréquence dans CCl_4 se trouve être la même que la fréquence de l'oscillation ν_3, se fend — à cause de la résonance de Fermi — en deux composantes: $\nu_3'=760$ cm^{-1} et $\nu_3''=788$ cm^{-1}. Alors, au lieu d'une bande $(\nu_1+\nu_4-\nu_4)$, il devrait exister

TABLEAU II

Largeurs des raies de CCl_4 aux diverses températures

N° de la plaque	$\nu_1 = 459$ cm^{-1}			$\nu_2 = 217$ cm^{-1}			$\nu_4 = 314$ cm^{-1}		
	20°	55°	65°	20°	55°	65°	20°	55°	65°
1	5.9		7.2	6.9		10.3	7.3		9.7
2			7.9			10.4			10.6
3	7.2			6.6			7.6		
4			7.9			10.0			10.0
5		7.1			9.1			9.7	
6		7.1			9.4			9.2	
7	6.4			6.7			7.2		
Moyennes	6.5	7.1	7.7	6.7	9.25	10.2	7.4	9.4	10.1
$t°$	20°		60°	20°		60°	20°		60°
Moyennes	6.5 ± 0.6		7.4 ± 0.3	6.7 ± 0.2		9.75 ± 0.2	7.4 ± 0.2		9.7 ± 0.5
Élargissement	0.9 ± 0.9			3.0 ± 0.4			2.3 ± 0.7		

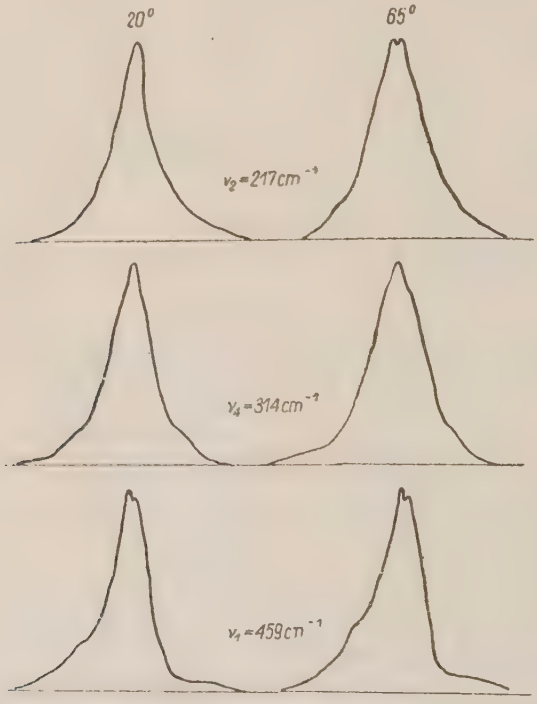


Fig. 2

deux bandes de différence $\nu_3' - \nu_4$ et $\nu_3'' - \nu_4''$ déplacées vers la raie ν_1 et par conséquent accessibles à l'observation [7].

L'intensité des bandes de différence devrait augmenter avec la température proportionnellement à la population du premier niveau excité de l'oscillation ν_4 (proportionnellement à $\exp(-h\nu_4/kT)$, pendant que l'intensité de la raie centrale ν_1 diminue. Autrement dit, avec l'augmentation de la température l'intensité des ailes de la raie devrait augmenter par rapport à la raie centrale. Dans le présent travail on a observé ce genre de modification du profil. Les résultats de ces observations apportent une preuve bien convaincante de l'apparition de ces bandes obtenues déjà par Rank et Van Horn [8].

Afin d'obtenir le profil approximatif des bandes de différence on a soustrait du profil expérimental de la raie ν_1 le profil résultant de l'addition des composantes isotopiques.

On a admis que:

1) les intervalles entre les composantes sont celles trouvées par Rank et Van Horn [8];

2) le profil de chaque composante est celui de la raie excitatrice d'une largeur choisie de telle sorte que les parties supérieures du profil théorique et du profil expérimental se confondent.

La figure 3 représente les résultats de cette soustraction. Les profils des bandes ν_3' et ν_3'' obtenus d'une des plaques sont placés plus bas à la

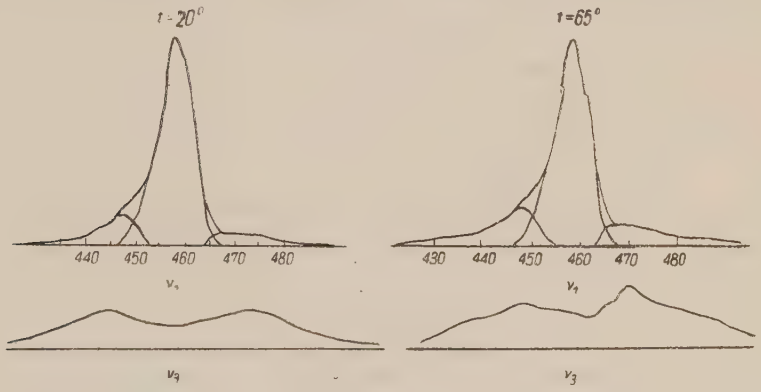


Fig. 3

même figure pour faciliter leur comparaison. Les maxima des bandes $\nu_3' - \nu_4$ et $\nu_3'' - \nu_4$ se trouvent à 447 cm^{-1} et 470-472 cm^{-1}. La distance observée des maxima s'accorde donc avec celle des bandes ν_3', ν_3'' à la température de 65°C. Mais elle est de quelques cm^{-1} plus courte à la température de 20°.

Cependant, l'intensité des bandes de différence a été surestimée, probablement parce que l'on a pris en considération les profils expérimentaux, non corrigés de la raie. Au lieu des valeurs théoriques, soit 20% environ de l'intensité de la raie ν_1 propre à la température de 20°C, et 30% environ à la température de 65°C, on a obtenu 24% et 37%.

L'auteur tient à rendre hommage au feu Professeur S. Pieńkowski pour ses suggestions qui ont facilité le choix du sujet de ce travail.

Elle tient aussi à remercier Mrs. les Professeurs J. Pniewski et T. Skaliński pour les précieux conseils, qu'ils lui ont prodigués au cours de ce travail.

INSTITUT DE PHYSIQUE EXPÉRIMENTALE DE L'UNIVERSITÉ DE VARSOVIE

OUVRAGES CITÉS

[1] J. J. Sobielman, Izv. A. N. SSSR **17** (1953), 554.
[2] P. A. Bajoulin et A. W. Rakov, D. A. N. SSSR **105** (1955), 54.
[3] B. J. Malychev, Izv. A. N. SSSR **16** (1950), No. 6.
[4] M. L. Sosiński, Izv. A. N. SSSR **18** (1954), 709.
[5] M. M. Souchtchinski, Izv. A. N. SSSR **11** (1947), No. 4.
[6] H. E. Sterin, Izv. A. N. SSSR **11** (1947), 348.
[7] G. Herzberg, *Infrared and Raman spectra*, New York 1949, p. 310.
[8] D. H. Rank et Van Horn, J. C. S. A. **33** (1946), 454.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 11, 1956

CHEMISTRY

Some Remarks on the Structure of Aniline Black, on the Basis of Ultraviolet Absorption Spectra

by

T. URBAŃSKI and K. SZYC-LEWAŃSKA

Communicated by T. URBAŃSKI on June 1, 1956

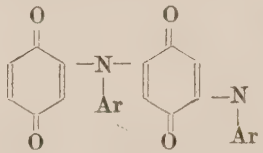
Two types of structural formulae of products of oxidation of aniline to aniline black — those of quinoneimine or phenazine — are generally accepted [1]. According to Willstätter [2] emeraldine, nigraniline, and pernigraniline possess structures consisting essentially of *p-quinoneimine rings* which are combined with benzene rings to form chains. The number of benzene rings decreases while that of quinoneimine rings increases with the progress of oxidation and deepening of colour of the product. Evidently the formation of the quinoneimine chains is the result of oxidation and condensation in *para* position.

This view of Willstätter has been criticised by Bucherer [3], who suggested another structure with *phenazine* rings. It has been supported by Green [4] who prepared "ungreenable" aniline black by oxidising pernigraniline of Willstätter in presence of aniline. The oxidation (in *ortho* position) leads to the formation of phenazine rings. This view has been critisised by Joffe and Metrikina [5]. According to them, "ungreenable" aniline black is simply pernigraniline.

It has been pointed out by a number of authors that the oxidation of aniline leads sometimes to the formation of red or brown anilinoquinone derivatives, i. e. the condensation occurs in *ortho* position. Thus, Schunck and Marchlewski [6] obtained a product, which they described as dianilino-quinone-monoanil, when oxidising aniline with hydrogen peroxide or ozone in a weakly acid medium. Similar results of the oxidation of aniline have been described by Bamberger and Tschirner [7], Szarvasy [8] and others [9].

Willstätter and Majima [10] pointed out that *para* condensation occurs when oxidation is carried out in a weakly acid medium in presence of some catalysts, such as vanadium chloride or ferric sulphate. Without catalysts an additional condensation can occur in the *ortho* position, and branched molecules can be formed.

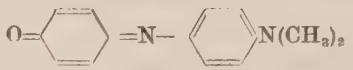
Suida [11], in his well-known work on reaction of benzoquinone and aniline, pointed out that benzoquinone and aniline may lead to the formation of dark coloured derivatives of anilinoquinone similar to aniline black in an oxidising medium. The products are polimeric derivatives of triphenylamine (A).



(A)

In the present work the authors made a number of experiments on ultraviolet absorption spectra of emeraldine (I), aniline black (pernigranilinę) (II), and "ungreenable" aniline black of Green (III) [4]— all in solution in sulphuric acid.

In order to compare with the spectra of the known compounds the spectra of solutions of model-compounds: anilinoquinone (IV), dianilinoquinone (V), phenazine (VI), saphranine (VII) and phenol blue (VIII) have been examined in sulphuric acid. Saphranine was taken as a typical phenazine dye. Phenol blue (N-dimethylindoaniline) has a structure composed of an indoaniline unit, similar to that of Willstätter's formula of aniline black.



(VIII)

Emeraldine and pernigraniline were prepared by oxidising aniline sulphate in 20% sulphuric acid solution, by means of chromic acid. The samples were purified by boiling with chloroform, washed with alcohol, then water; next, boiling with 10% sulphuric acid and finally washed with water, aqueous ammonia, and water.

Ungreenable aniline black was prepared according to Green [4], by oxidising pernigraniline with chromic acid in presence of aniline.

Anilinoquinone and dianilinoquinone were prepared according to Suida [11]. Phenazine and saphranine were commercial grade products purified by crystallisation from the appropriate solvents. Phenol blue was prepared by condensation of *p*-nitrosodimethylaniline with phenol. It was isolated and purified in the form of a complex salt with zinc chloride.

The results of spectrophotometric analysis are collected in Table I and on diagrams, Figs. 1-8. To calculate the extinction coefficients of emeraldine and pernigraniline the molecular weights were taken according to the formulae of Willstätter. In the case of "ungreenable" black — the molecular weight according to Green.

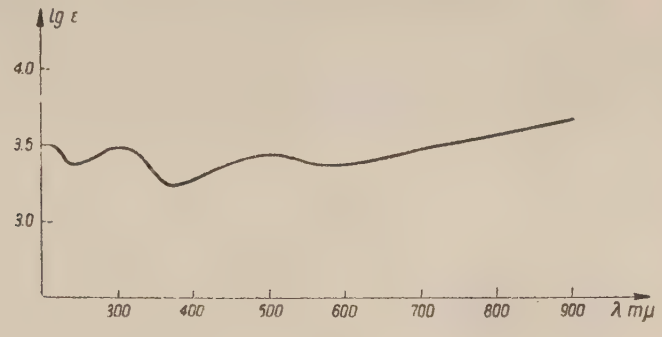


Fig. 1

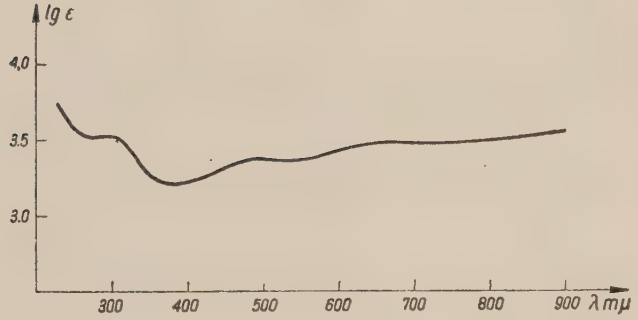


Fig. 2

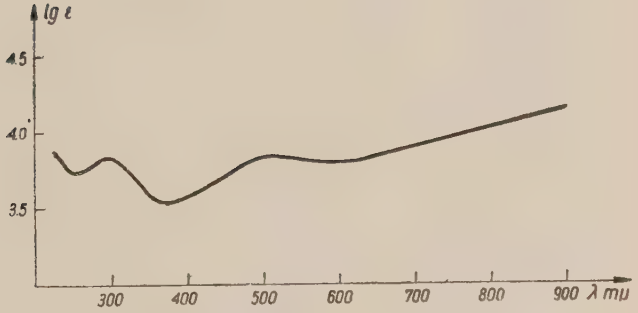


Fig. 3

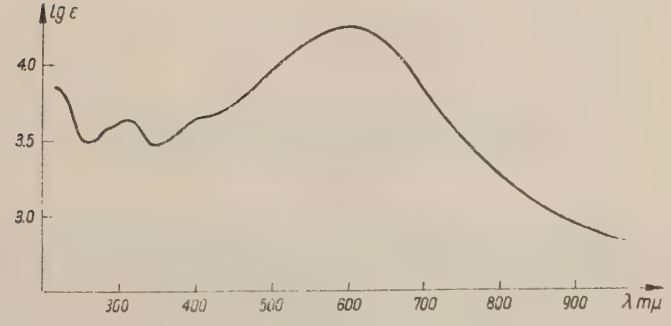


Fig. 4

There is evident similarity between the spectra of emeraldine and aniline black (pernigraniline and ungreenable black) on the one side, and dianilinoquinone on the other. On the other hand, a clear difference exists between the spectra of emeraldine or aniline black and phenazine

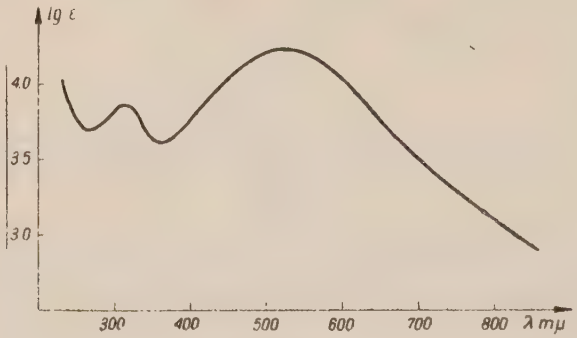


Fig. 5

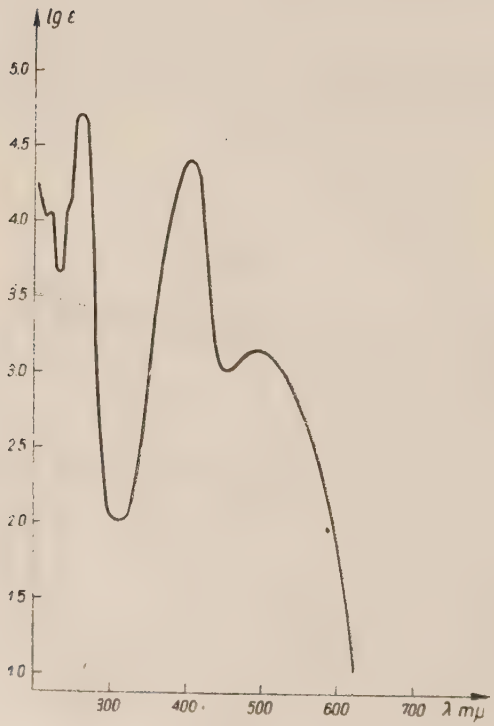


Fig. 6

or saphranine. Also the spectrum of phenol blue differs very much from the spectrum of emeraldine and aniline black.

This would suggest that emeraldine, nigraniline, pernigraniline and ungreenable aniline black may all possess a structure similar to that

of dianilinoquinone. The *p*-quinoneiminine structure of Willstätter, or the phenazine structure given by Bucherer and Green cannot be confirmed by the absorption spectra of emeraldine and aniline black.

On the basis of preliminary analytical data obtained for purified

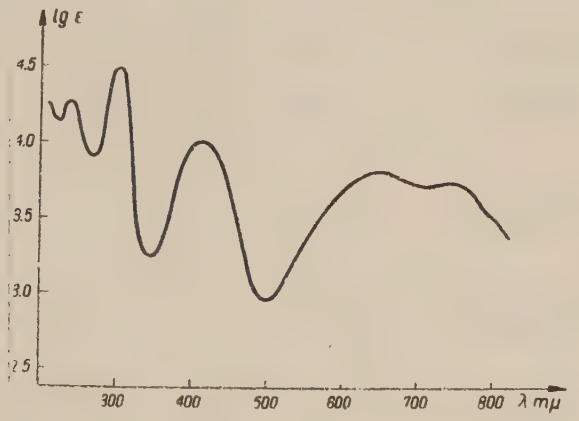


Fig. 7

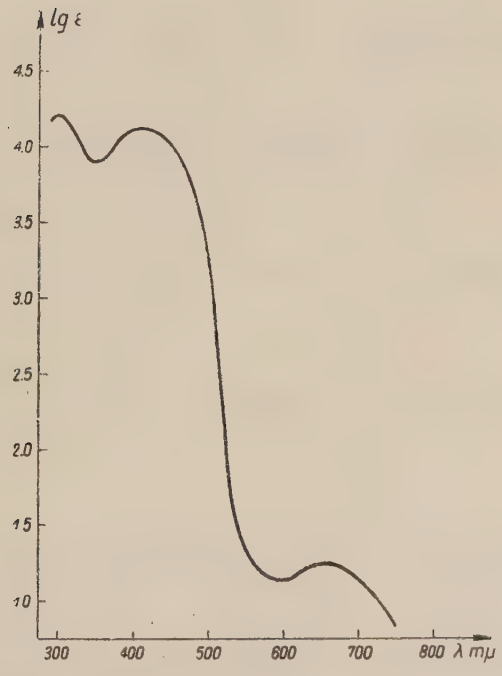


Fig. 8

pernigraniline, prepared by oxidation of aniline sulphate with chromic acid, a relatively high oxygen content should be admitted.

One sample of aniline black gave an empirical formula near to $C_{30}H_{20}O_6N_4$.

TABLE I

No. of curve	Substance	Colour of the solution in sulphuric acid	Absorption curve			
			max.		min.	
			mμ	log ε	mμ	log ε
1	Emeraldine	magenta	300	3.83	245	3.72
			510	3.83	370	3.50
					570	1.78
2	Aniline black (pernigraniline)	bluish-green	290	3.52	275	3.51
			520	3.41	375	3.23
			670	3.50	740	3.48
3	Aniline black (ungreenable)	red *)	300	4.00	340	3.86
			500—510	3.94	370	3.72
4	Anilinoquinone	blue	312	3.63	345	3.48
			(405) **)	(3.67)	(365) **)	(3.47)
			590	4.23	425	3.61
5	Dianilinoquinone	red	308	3.89	270	3.68
			520	4.23	345	3.61
6	Phenazine	orange-red	260	4.74	240	3.75
			418	4.42	318	2.00
			500—510	3.09	450	2.99
7	Saphranine	green	240	4.28	220	4.13
			300	4.50	264	3.90
			420	4.05	344	3.20
			640	3.84	510	2.95
8	Phenol blue	orange	300	4.15	350	3.90
			415	4.13	585	1.08
			650	1.18		

Analysis:

$C_{30}H_{20}O_6N_4$ required — 67.7% C, 3.7% H, 10.5% N
found — 67.9% C, 4.4% H, 10.9% N, 0.1% Cr.

We do not attempt, for the time being, to suggest any definite structure of aniline black, but a structure in principle similar to (A) appears to be probable.

*) The solution contained a considerable quantity of undissolved suspended particles.

**) A bend, without a definite maximum.

Further experiments are being carried out. The results, which include infra-red absorption spectra of emeraldine, aniline black, anilinoquinones, and phenazine as well as analytical data will be reported in due course.

The authors are much indebted to D. Ciecierska and K. Gorczyńska, for carrying out experiments on ultraviolet absorption spectra. Our thanks are also due to W. Daniewski jr. and M. Witanowski for preparing a sample of phenol blue.

INSTITUTE OF ORGANIC SYNTHESIS, POLISH ACADEMY OF SCIENCES
TECHNICAL MILITARY COLLEGE

REFERENCES

[1] P. Karrer, *Lehrbuch der Organischen Chemie*, G. Thieme, Stuttgart, 1954.
L. F. Fieser, M. Fieser, *Organic chemistry*, Heath and Co, Boston, 1950.
K. Venkataraman, *The chemistry of synthetic dyes*, Academic Press Inc., New York, 1952, p. 777.
[2] R. Willstätter, et al., Ber. **40** (1907), 2665; **42** (1909), 2147, 4118; **43** (1910), 2976; **44** (1911), 2162.
[3] H. Bucherer, Ber. **40** (1907), 3412.
[4] A. G. Green et al., J. Chem. Soc. **97** (1910), 2388; **101** (1912), 1117; Ber. **44** (1911), 2570; **46** (1913), 33; J. Soc. Dyers Cols. (1913), 338.
[5] J. S. Joffe, R. M. Metrikina, J. Russ. Chem. Soc. **62** (1930), 1101, 1115.
[6] E. Schunk, L. Marchlewski, Ber. **25** (1892), 3574.
[7] E. Bamberger, Tschirner, Ber. **31** (1898), 1522.
[8] E. Ch. Szarvasy, J. Chem. Soc. **31** (1898), 1522.
[9] e. g. Börnstein, Ber. **34** (1901), 1268.
[10] R. Willstätter, R. Majima, Ber. **43** (1910), 2588.
[11] H. Suida et al., Ann. **416** (1918), 113, 164, 181.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 11, 1956

CHEMISTRY

On the Production of Sodium Pyrophosphite ($Na_2H_2P_2O_5$) by Pyrolysis of Disodium Hypophosphate ($Na_2H_2P_2O_6$)

by

J. H. KOLITOWSKA

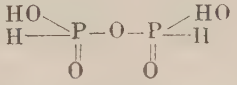
Presented by T. MIŁOBĘDZKI on July 7, 1956

It is generally known that disodium pyrophosphite $Na_2H_2O_2P_5$ is obtained after the method elaborated by Amat [1] — i. e. by heating monosodium phosphite NaH_2PO_3 for 5 hours in an electric oven at 160°C. At this temperature, condensation of the phosphite occurs with simultaneous evolution of water. The reaction is quantitative. E. Thile and D. Heinz [2] published a paper on hydrolysis of trivalent phosphorus halides. They found that chromatographic and volumetric (iodometric) examinations revealed that almost the entire amount of PCl_3 hydrolysed in an $NaHCO_3$ solution, either devoid of or containing monosodium phosphite NaH_2PO_3 (in amounts corresponding to the quantity of PCl_3 used in the reaction), is transformed into pyrophosphite $[H_2P_2O_5]^{2-}$.

Thile and Heinz believe that the production of pyrophosphite is due to condensation by splitting off HCl from the P—Cl bond and from the weakly dissociating OH group, according to the following equation:

$$P{-}Cl + HO{-}P \rightarrow HCl + P{-}O{-}P.$$

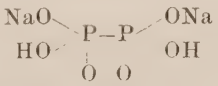
It is known that the structure of the pyrophosphorous acid $H_2[H_2P_2O_5]$ is as follows:



The Raman method reveals lines corresponding to the bonds (P—H) and (P—O—P) [3], and the molecular weight, determined by the depression of the Glauber salt transition point, reaches the value of 183 instead of the theoretical 190 [4].

Pyrophosphorous acid is easily hydrolysed in acid or alkaline solution to $2H_2[HPO_3]$. In an $NaHCO_3$ solution it is not oxidised by iodine to orthophosphates. Prior to oxidisation with iodine in an $NaHCO_3$ solution, it must be hydrolysed to $2H_2[HPO_3]$.

In the course of my experiments, here reported on, I became convinced that sodium pyrophosphite $Na_2H_2P_2O_5$ is produced when disodium hypophosphate $Na_2H_2P_2O_6$ is heated at a temperature of 230°C. At this temperature, oxidation-reduction takes place, and thus polyphosphates are produced simultaneously with pyrophosphite. The structure of disodium hypophosphate is as follows:



the hypophosphate thus containing the P—P bond. The $Na_2H_2P_2O_6$ transformation products, on the other hand, the precipitated barium polyphosphates, and also the soluble barium pyrophosphite, have the P—O—P bond.

A solution of the pyrophosphite obtained, subjected to hydrolysis by heating with acids, aquires the capacity to reduce iodine in an $NaHCO_3$ solutions; the amount of P^{3+}, determined iodometrically, is in almost complete agreement with the amount of phosphorus as determined by the gravimetric method.

On the other hand, iodometric determination reveals but an insignificant P^{3+} content in the original solution obtained by dissolving $Na_2H_2P_2O_6$ heated to 230°C.

Description of experiments

$Na_2H_2P_2O_6.6H_2O$, crystallised and analysed, was obtained after the method of J. Cavalier and E. Cornec [5]. Anhydrous samples c. (0.85 g.) were heated in an air-bath to a temperature of 230°C. At this temperature, the substance increases in volume, becomes loose and gives off the smell associated with PH_3 traces. The weight decrement (c. 1.3 per cent) is not large after the termination of the reaction. I dissolved the loose substance in a 250 c. c. calibrated flask. The substance dissolves readily and phosphine can scarcely be detected by smell. *pH* of the solution is about 6. Addition of an 0.1 *n* solution of $AgNO_3$ to samples of the solution produce slight white turbidity. After some time, the hardly perceptible precipitate grows black, indicating gradual hydrolysis of the pyrophosphite produced.

In sample I (50 c. c.) of the original solution, I determined the amount of P^{3+} iodometrically, in an $NaHCO_3$ solution.

Sample II of the original solution (100 c. c.) was acidified with 1 c. c. of 1 *n* HCl and mixed with 48 c. c. of 2.5 per cent $BaCl_2$ solution. Absence of a precipitate indicated absence of long-chain polyphosphates ("hexamethaphosphate" [6]). When the solution was neutralised with 1 *n* NaOH in the presence of phenolphthalein, barium salts are precipitated and were subsequently filtered off. The precipitate and the filtrate were tested for P^{3+} content.

a. Analysis of the barium salts precipitate.

The barium salts were dissolved in hydrochloric acid and Ba^{2+} kation was eliminated by filtration as $BaSO_4$ obtained by treating with H_2SO_4. The filtrate, acidified with a few c. c. of 2 *n* H_2SO_4 was kept for a few hours in a hot water bath. Subsequently, the solution was neutralised with NaOH, and P^{3+} was determined iodometrically, in the presence of $NaHCO_3$.

b. Analysis of the filtrate after barium salts precipitation.

Ba^{2+} kation was eliminated by filtration as $BaSO_4$ obtained by treating with H_2SO_4. The filtrate, acidified with a few c. c. of 2 *n* H_2SO_4, was kept for a few hours in a hot water bath. Subsequently, the solution was neutralised with 1 *n* NaOH and P^{3+} was determined iodometrically in the presence of $NaHCO_3$.

Sample III of the original solution (90 c. c.) was mixed with 48 c. c. of 2.5 per cent $BaCl_2$ in an alkaline solution. The barium salts were filtered off and the total phosphorus was determined in both the barium salt precipitate and the filtrate.

c. Analysis of the barium salt precipitate.

The barium salts were dissolved in 2 *n* HCl, and kation Ba^{2+} was filtered off as $BaSO_4$; the filtrate was repeatedly evaporated with aqua regia and phosphorus was determined as $Mg_2P_2O_7$.

d. Analysis of the filtrate after barium salts precipitation.

The filtrate was repeatedly evaporated with aqua regia and the phosphorus precipitated with ammonium molybdate reagent. Phosphorus was determined as $Mg_2P_2O_7$. These results are recorded in Table I.

When the hypophosphate $Na_2H_2P_2O_6$ is heated for a longer time and to a higher temperature, long-chain polyphosphates ("hexamethaphosphate") are produced, and a part of the barium salts is precipitated also in an acid solution. At the same time, a side reaction sets in, viz., oxidation of hypophosphate (P^{+4}) to polyphosphates (P^{+5}) with concomitant evolution of hydrogen. Hence the reduced yield of pyrophosphites.

These results are recorded in Table II.

Conclusions

At a temperature of 230°C, $Na_2H_2P_2O_6$ undergoes complete internal transformation which may be represented schematically by the following simplified equation:

$$2Na_2H_2P_2^{4+}O_6 \rightarrow Na_2H_2P_2^{3+}O_5 + Na_2H_2P_2^{5+}O_7.$$

The simplification consists in disregarding the further condensation of sodium pyrophosphate to higher polyphosphates, and also the slight disproportionation of the substrate from P^{4+} to P^{3-} and P^{5+}.

TABLE I

Weight of $Na_2H_2P_2O_6$ sample in g.	Total time of heating in h.	Time of heating at 230° C in h.	P content in barium salts from acid solutions. per cent.	P content in barium salts from alkaline solutions. per cent.	P in filtrate after barium salts. per cent.	Total P per cent.	Weight decrement after heating. per cent.	P^{3+} content in barium salts from acid solution after hydrolysis. per cent.	P^{3+} content in barium salts from alkaline solution after hydrolysis. per cent.	P^{3+} content in filtrates after barium salts after hydrolysis. per cent.	P^{3+} in the original non-hydrolysed solution. per cent.	P^{3+} in the original solution after hydrolysis. per cent.
0.8574	1^{15}	0^{30}	none	49.43	43.99	93.42*)	1.26	none	3.05	42.66	1.20	—
0.8470	1^{05}	0^{25}	none	55.63	42.70	98.33	1.23	none	3.72	40.76	2.13	—

TABLE II

Weight of $Na_2H_2P_2O_5$ sample in g.	Total time of heating in h.	Time of heating at 290° C. in h.	P content in barium salts from acid solutions. per cent.	P content in barium salts from alkaline solutions. per cent.	P in filtrate after barium salts. per cent.	Total P per cent.	Weight decrement after heating. per cent.	P^{3+} content in barium salts from acid solution after hydrolysis. per cent.	P^{3+} content in barium salts from alkaline solution after hydrolysis. per cent.	P^{3+} content in filtrates after barium salts after hydrolysis. per cent.	P^{3+} in the original non-hydrolysed solution. per cent.	P^{3+} in the original solution after hydrolysis. per cent.
0.8581	2^{00}	1^{30}	26.31	39.81	27.39	93.51	2.04	1.42	1.95	24.91	3.30	—
0.8034	1^{35}	0^{35}	30.80	27.69	36.90	95,39	—	—	—	—	5.99	33.96
0.7860	1^{40}	0^{30}	32.10	20.99	43.86	96.95	—	—	—	—	—	—

*) In spite of thorough washing with hot water, $BaSO_4$ precipitates absorbed some P_2O_5, the presence of which was proved by boiling the $BaSO_4$ precipitate with an ammonium molybdate reagent and with nitric acid. This accounts for the phosphorus deficit.

Summary

It has been established for the first time that sodium pyrophosphite $Na_2H_2P_2O_5$ is produced by subjecting disodium hypophosphate $Na_2H_2P_2O_6$ to oxidation-reduction at a temperature of 230° C. Oxidation-reduction is connected with the simultaneous transformation of the P—P bond existing in the hypophosphate in the form P—O—P present in the pyrolysis products, the pyrophosphite and the polyphosphates simultaneously obtained.

TECHNOLOGICAL INSTITUTE, DEPARTMENT OF ANORGANIC CHEMISTRY
POLISH ACADEMY OF SCIENCES

REFERENCES

[1] Amat, C. R. hebd. Seances Acad. Sci. **106** (1888), 1400; **108** (1889), 10563; **110** (1890), 191.

[2] E. Thilo u. D. Heinz, Z. anorg. u. allg. Chem. **281** (1955), 303.

[3] Baudler (quoted in [2]).

[4] K. F. Jahr u. R. Kubens, Z. Elektrochem. Ber. Bunsenges. physik. Chem. **56** (1952), 65.

[5] J. Cavalier et E. Cornec, Bull. Soc. Chim. **5** (1909), 1058.

[6] W. Dewald, H. Schmidt, Z. analyt. Chem. **137** (1952), 178.